Recitation 3 - Consumer Theory *

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Review of Consumer Theory Axioms

Recall from L4 that we generally assume five axioms of consumer preferences in order to represent decisions with well-behaving utility functions and demand curves. Understanding these five axioms deeply (along with their mathematical implications) will be important to solving many problems in this class (including problems in PS2).

1. **Completeness** – consumers have preference orderings over any two bundles of goods, every consumption bundle lies on some indifference curve
2. **Transitivity** – consumers are *consistent*
3. **Continuity** – indifference curves are smooth
4. **Non-satiation** – consumers always prefer to have more of any given good (when holding the amount of other goods constant)
5. **Diminishing MRS** – indifference curves are convex and utility function U(.) is concave

All of these have mathematical (and graphical) implications that you should remember, a few particularly useful ones are:

- For a concave function f, we have that for all $0 \leq \alpha \leq 1$ and any x, y, it follows that $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$
- MRS of x for y = $\frac{MU_x}{MU_y}$. MRS can be calculated along an indifference curve relative to some initial bundle (x_1, y_1), so you can calculate the MRS of x for y by taking differences of the quantities $|\frac{\Delta y}{\Delta x}|$ between two bundles on the same indifference curve (recall the cold brew and sushi example from class)

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Duality and Demand

Recall from R1 that duality is the notion that optimization problems can be viewed from either of two perspectives, the *primal problem* and the *dual problem*. We turn our attention now to utility maximization (subject to a budget constraint) as the *dual problem* of expenditure minimization (holding constant a certain utility level). In the next four sections, we will work with an illustrative utility function $u(x, y) = \frac{3}{4}\ln x + \frac{1}{4}\ln y$ and walk through four related steps:

1. Solving the primal problem (utility maximization) to get Marshallian demand
2. Plugging the solutions of the primal problem back into $u(.)$ to get the indirect utility function
3. Solving the dual problem (expenditure minimization) to get Hicksian demand
4. Plugging the solutions of the dual problem back into $p \cdot x$ to get the expenditure function

1. Solving the primal problem to get Marshallian demand

The consumer is trying to maximize their utility subject to a budget constraint:

$$\max_{x,y} \frac{3}{4}\ln x + \frac{1}{4}\ln y \quad \text{subject to} \quad p_x x + p_y y = I$$

Note: we could have also maximized $3\ln x + \ln y$ *subject to the same constraint. Why?*

The first step towards maximizing this utility function is to write the Lagrangian:

$$\mathcal{L}(x, y, I) = \frac{3}{4}\ln x + \frac{1}{4}\ln y + \lambda(I - p_x x - p_y y).$$

This gives us three FOCs:

$$\begin{aligned} \mathcal{L}_x &: \ \frac{3}{4x} - \lambda p_x = 0 \\ \mathcal{L}_y &: \ \frac{1}{4y} - \lambda p_y = 0 \\ \mathcal{L}_\lambda &: \ p_x x + p_y y - I = 0. \end{aligned}$$

Solving this system gives us the Marshallian demand functions that are functions of the price vector (p_x and p_y) and the income I:

$$x^*(p_x, p_y, I) = \frac{3I}{4p_x}$$
$$y^*(p_x, p_y, I) = \frac{I}{4p_y}.$$

2. Plugging the solutions of the primal problem back into $u(.)$ to get the indirect utility function

Plugging $x^*(p_x, p_y, I)$ and $y^*(p_x, p_y, I)$ into $u(x, y)$ gives us the maximized utility function. We refer to this as the indirect utility function (or value function):

$$\begin{aligned}\underbrace{V(p_x, p_y, I)}_{\text{function of parameters}} &= u(x^*(p_x, p_y, I), y^*(p_x, p_y, I)) \\ &= \frac{3}{4}\ln\left(\frac{3I}{4p_x}\right) + \frac{1}{4}\ln\left(\frac{I}{4p_y}\right).\end{aligned}$$

Note: What if I want the expenditure function $e(p, \overline{u})$? I can simply invert the indirect utility function! Alternatively, you can follow steps 3 and 4 ahead.

$$\begin{aligned}\overline{u} &= \frac{3}{4}\ln\left(\frac{3I}{4p_x}\right) + \frac{1}{4}\ln\left(\frac{I}{4p_y}\right) \\ &= \frac{1}{4}\ln\left(\frac{27I^4}{4^4 p_x^3 p_y}\right) \\ \implies e^{\overline{u}4} &= \left(\frac{I}{4}\right)^4 \frac{3^3}{p_x^3 p_y}\end{aligned}$$

Rearranging for I gives the expenditure function:

$$E(p_x, p_y, \overline{u}) = 4e^{\overline{u}}\left(\frac{p_x^3 p_y}{3^3}\right)^{\frac{1}{4}}$$

3. Solving the dual problem (expenditure minimization) to get Hicksian demand

This time, the consumer is trying to minimize their expenditure subject to a utility constraint:

$$\min_{x,y} p_x x + p_y y \quad \text{subject to} \quad \frac{3}{4}\ln x + \frac{1}{4}\ln y = \overline{u}.$$

We can write the Lagrangian as

$$\mathcal{L}(x,y,V) = p_x x + p_y y + \lambda^D \left(\overline{u} - \frac{3}{4}\ln x - \frac{1}{4}\ln y\right).$$

This gives us three FOCs:

$$\begin{aligned}
\mathcal{L}_x :&\ p_x - \lambda^D \frac{3}{4x} = 0 \\
\mathcal{L}_y :&\ p_y - \lambda^D \frac{1}{4y} = 0 \\
\mathcal{L}_{\lambda^D} :&\ \frac{3}{4}\ln x + \frac{1}{4}\ln y - \overline{u} = 0.
\end{aligned}$$

Solving this system gives us the Hicksian demand functions that are functions of the price vector (p_x and p_y) and the utility level $\overline{u}$:

$$\begin{aligned}
x^h(p_x, p_y, \overline{u}) &= e^{\overline{u}} \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} \\
y^h(p_x, p_y, \overline{u}) &= e^{\overline{u}} \left(\frac{p_x}{3p_y}\right)^{\frac{3}{4}}.
\end{aligned}$$

4. Plugging the solutions of the dual problem back into $p \cdot x$ to get the expenditure function

Plugging $x^h(p_x, p_y, \overline{u}) = e^{\overline{u}} \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}}$ and $y^h(p_x, p_y, \overline{u}) = e^{\overline{u}} \left(\frac{p_x}{3p_y}\right)^{\frac{3}{4}}$ into $p_x x + p_y y$ gives us the minimized expenditure function:

$$\begin{aligned}
E(p_x, p_y, \overline{u}) &= p_x x^h + p_y y^h \\
&= p_x e^{\overline{u}} \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} + p_y e^{U} \left(\frac{p_x}{3p_y}\right)^{\frac{3}{4}} \\
&= e^{\overline{u}} p_x^{\frac{3}{4}} p_y^{\frac{1}{4}} \left(3^{\frac{1}{4}} + 3^{-\frac{3}{4}}\right) \\
&= \frac{e^{\overline{u}} p_x^{\frac{3}{4}} p_y^{\frac{1}{4}} \left(3^1 + 3^0\right)}{3^{\frac{3}{4}}} \\
&= \frac{4e^{\overline{u}} p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}}} \\
&= 4e^{\overline{u}} \left(\frac{p_x^3 p_y}{3^3}\right)^{\frac{1}{4}}
\end{aligned}$$

Notice that this last expression is the same as that obtained by inverting the indirect utility function in step 2. In general, if you have the functional form of the indirect utility function $V(p, I) = \overline{u}$, you can invert it to get the functional form of the expenditure function $E(p, \overline{u}) = I$ and vice versa. Note that while in these notes we used upper case $V(.)$ and $E(.)$ to denote the indirect utility function and the expenditure function to avoid confusion with Euler's number e, they are often denoted with lower case letters $v(.)$ and $e(.)$.

Deriving Roy's Identity

We will now turn use the first order conditions (FOCs) to derive a famous relationship between Marshallian demand and the indirect utility function, known as Roy's Identity:

$$\underbrace{x^m(p, I)}_{\text{Marshallian demand}} = -\frac{\frac{\partial V}{\partial p_x}}{\frac{\partial V}{\partial I}}$$

At the optimum for a general indirect utility function $V(p_x, p_y, I)$:

$$V(p_x, p_y, I) = u(x^*, y^*) + \lambda^*(I - p_x x^* - p_y y^*)$$

Taking the derivate with respect to p_x, bearing in mind that x^* and y^* are themselves functions of the price vector p_x and p_y, gives:

$$\begin{aligned} \frac{\partial V}{\partial p_x} &= \underbrace{\left(\frac{\partial u}{\partial x} - \lambda^* p_x\right)}_{=\mathcal{L}_x=0} \frac{\partial x^*}{\partial p_x} + \underbrace{\left(\frac{\partial u}{\partial y} - \lambda^* p_y\right)}_{=\mathcal{L}_y=0} \frac{\partial y^*}{\partial p_x} + \underbrace{(I - p_x x^* - p_y y^*)}_{=\mathcal{L}_\lambda=0} \frac{\partial \lambda^*}{\partial p_x} - \lambda^* x^* \\ &= -\lambda^* x^* \end{aligned}$$

Taking the derivate with respect to I, bearing in mind that x^* and y^* are themselves functions of income I, gives:

$$\begin{aligned} \frac{\partial V}{\partial I} &= \underbrace{\left(\frac{\partial u}{\partial x} - \lambda^* p_x\right)}_{=\mathcal{L}_x=0} \frac{\partial x^*}{\partial I} + \underbrace{\left(\frac{\partial u}{\partial y} - \lambda^* p_y\right)}_{=\mathcal{L}_y=0} \frac{\partial y^*}{\partial I} + \underbrace{(I - p_x x^* - p_y y^*)}_{=\mathcal{L}_\lambda=0} \frac{\partial \lambda^*}{\partial I} + \lambda^* \\ &= \lambda^* \end{aligned}$$

Taking the ratio, we get Roy's identity:

$$x^m(p_x, p_y, I) = -\frac{\frac{\partial V(p_x, p_y, I)}{\partial p_x}}{\frac{\partial V(p_x, p_y, I)}{\partial I}} = -\frac{-\lambda^* x^*}{\lambda^*} = x^*$$

Note that the expression we got to find the denominator for Roy's identity is the result we saw in class that the envelope theorem tells us that the derivative of the indirect utility function is equal to the derivative of this Lagrangian at the optimum, $\frac{\partial V}{\partial I} = \frac{\partial \mathcal{L}}{\partial I}$. Let's return to the indirect utility function we found earlier and find Marshallian demand using Roy's identity:

$$V(p_x, p_y, I) = \frac{3}{4} \ln\left(\frac{3I}{4p_x}\right) + \frac{1}{4} \ln\left(\frac{I}{4p_y}\right).$$

Taking the derivatives with respect to p_x and I we get:

$$\frac{\partial V}{\partial p_x} = -\frac{3}{4p_x}$$
$$\frac{\partial V}{\partial I} = \frac{1}{I}$$

$$\implies x^m(p_x, p_y, I) = \frac{3I}{4p_x}$$

Deriving Shephard's Lemma

We will now turn use the first order conditions (FOCs) to derive a famous relationship between Hicksian demand and the expenditure function, known as Shephard's lemma:

$$x^h(p_x, p_y, \overline{u}) = \frac{\partial E(p_x, p_y, \overline{u})}{\partial p_x}$$

At the optimum for a general expenditure function $E(p_x, p_y, \overline{u})$:

$$E(p_x, p_y, \overline{u}) = p_x x^h + p_y y^h + \lambda^h(\overline{u} - u(x^h, y^h))$$

where x^h, y^h, and λ^h are the solutions to the dual problem.

Taking the derivate with respect to p_x, bearing in mind that x^h and y^h are themselves functions of the price vector p_x and p_y, gives:

$$\begin{aligned}
\frac{\partial E}{\partial p_x} &= \frac{\partial p_x}{\partial p_x}x^h + p_x\frac{\partial x^h}{\partial p_x} + p_y\frac{\partial y^h}{\partial p_x} + \overline{u}\frac{\partial \lambda^h}{\partial p_x} - \frac{\partial \lambda^h}{\partial p_x}u(x^h, y^h) - \lambda^h\frac{\partial u}{\partial y^h}\frac{\partial y^h}{\partial p_x} - \lambda^h\frac{\partial u}{\partial x^h}\frac{\partial x^h}{\partial p_x} \\
&= x^h + \underbrace{\left(p_x - \lambda^h\frac{\partial u}{\partial x^h}\right)}_{=\mathcal{L}_x=0}\frac{\partial x^h}{\partial p_x} + \underbrace{\left(p_y - \lambda^h\frac{\partial u}{\partial y^h}\right)}_{=\mathcal{L}_y=0}\frac{\partial y^h}{\partial p_x} + \underbrace{\left(\overline{u} - u(x^h, y^h)\right)}_{=\mathcal{L}_\lambda=0}\frac{\partial \lambda^h}{\partial p_x} \\
&= x^h
\end{aligned}$$

Returning to the expenditure function we found earlier, we can find the Hicksian demand function using Shephard's lemma:

$$E(p_x, p_y, \overline{u}) = \frac{4e^{\overline{u}}p_x^{\frac{3}{4}}p_y^{\frac{1}{4}}}{3^{\frac{3}{4}}}$$

Taking the derivative with respect to p_x, we get:

$$x^h(p_x, p_y, \overline{u}) = e^{\overline{u}}\left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}}$$