1 (18.03 $\setminus$ 18.06) ASE Reference

This reference is intended as a condensed reference to material in 18.03 not covered in 18.06 (although I do review some stuff from 18.06). This is mainly intended as a resource for passing the ASE, so I'm intentionally brief and sometimes emphasize practice over theory. I've divided these notes into two sections: the first which covers essential material for the exam, and the second which covers other material necessary for the packet, but unlikely to show up on the exam.

Before we get into the actual math, here are some tips

- From my experience with practice exams and the actual ASE (IAP 2025), the following problems will almost certainly occur:
  - Analyzing a (possibly nonlinear) DE with isoclines
  - Solving an equation of the form $z^n + k = 0$ where $n \geqslant 3$ (roots of unity)
  - Sketching the phase line, phase portrait, and or bifurcation diagram of some autonomous equation
  - Solving a "miscellaneous" DE (probably with integrating factors or standard calculus)
  - Expressing a sum of a sine and cosine or a "complex" complex number as a simple sine/cosine with a phase shift.
  - Solving an inhomogeneous linear DE with constant coefficients
  - Analyzing the gain of a damped linear system
  - Using the Fourier series to solve a DE
  - Finding the Fourier series of some simple periodic function
  - Solving a 2x2 system of linear differential equations with constant coefficients (eigenvectors, eigenvalues, phase portrait)
  - Finding the critical points of some nonlinear system and analyzing their stability with linearization
  - Using separation of variables to solve the heat equation
- For more thorough notes, the official course notes (https://math.mit.edu/~jorloff/suppnotes/suppnotes03/) and Professor Jörn Dunkel's notes (https://math.mit.edu/~dunkel/Teach/18.03/2018_CourseNotes.pdf) are good.
- There are a few practice exams at https://www.studocu.com/en-us/course/massachusetts-institute-of-technology/differential-equations/744206
- However you acquire the textbook, make sure to get a copy of Elementary Differential Equations *with Boundary Value Problems*, which has the correct page numbers and material on Fourier series.

Contents

2 Essential Material

2.1 Definitions

ordinary DE does not contain partial derivatives.

nth-order DE contains up to the nth derivative of its variables.

IVP initial value problem.

autonomous DEs of the form $\frac{dy}{dx} = f(y)$. For example, in $\frac{dy}{dx} = y^2 + 2y + 1$, x does not appear on the RHS.

linear nth order DE can be expressed in the form $\sum_0^n A_n(x)y^{(n)} = f(x)$; the coefficients of the $y^{(i)}$ are all functions solely of x. A linear DE is said to be **homogenous** if $f(x) = 0$.

complementary equation The complementary equation of an inhomogeneous linear DE $\sum_0^n A_n(x)y^{(n)} = f(x)$ is simply the equation $\sum_0^n A_n(x)y^{(n)} = 0$, i.e. set the right hand side to 0 instead of some function of x.

2.2 Separable ODEs

If the DE can be written in the form:

$$g(y)y' = f(x)$$

Integrating both sides and solving for y solves the equation.

2.3 Integrating Factors

If the DE can be written in the form:

$$y' + P(x)y = Q(x)$$

Multiply both sides of the equation by the integrating factor $e^{\int P(x)dx}$ to yield:

$$e^{\int P(x)dx}y' + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

Since left side is the derivative of $e^{\int P(x)dx}y$ (Product Rule), integrating both sides and solving for y solves the equation.

2.4 Existence and Uniqueness For First-Order DEs

For an IVP:

$$y' = f(x, y),\ y(x_0) = y_0$$

if $f(x, y)$ and $f_y(x, y)$ are continuous on some rectangle that contains (x_0, y_0), then there exists h such that the IVP has a unique solution on $(x_0 - h, x_0 + h)$. Furthermore, distinct solutions to a well-behaved DE like this cannot intersect (as that would mean $f(x, y)$ would have two different values for a single (x, y)).

2.5 Isoclines

An isocline for a DE $y' = f(x, y)$ is a curve of the form $f(x, y) = c$. A nullcline is an isocline with $c = 0$. We can often get the portrait of the solutions of a DE by drawing a slope field using a few values of c.

2.6 Autonomous DEs and Bifurcation

Recall that an autonomous DE has the form:

$$\frac{dx}{dt} = f(x) \tag{1}$$

that is, x' can be expressed solely as a function of x. Our task will mainly be to determine properties of solutions of these equations, rather than to find actual solutions.

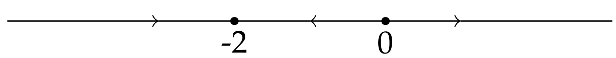
2.6.1 Critical Points

Firstly, notice that each solution c to $f(x) = 0$ gives us a constant solution to (1): $x = c$. These c values are called **critical points**. A critical point is **stable** if, loosely speaking, a solution near the critical point stays near the critical point.

Let's look at an example. Suppose we have the autonomous DE (1E-1a from the homework packet):

$$x' = x^2 + 2x \tag{2}$$

We can immediately spot two critical points $x = -2$ and $x = 0$, which solve $x^2 + 2x = 0$. Now, let's examine their stability using a **phase diagram**:



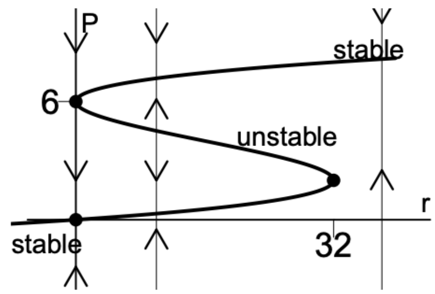
Firstly, the two critical points are marked on the phase diagram. Secondly, the sign of x' is also marked using arrows showing the "direction of motion" of x. This diagram tells us that when $x < -2$ or $x > 0$, x' is positive. When $-2 < x < 0$, x' is negative. This is enough information to determine stability of the critical points. Notice that both arrows adjacent to $x = -2$ are pointing towards it. This means -2 is a stable critical point. If x is greater than -2 (but less than 0), x' is negative, so x is going to decrease towards -2. Meanwhile, if $x < -2$, the x' is positive, so x is going to increase towards -2. On the other hand, both arrows adjacent to $x = 0$ are pointing away, which means it is unstable. The last case is where both arrows adjacent to a critical point are pointing the same way; this is called semistability. If you start on the correct side of the critical point, you'll end up there, otherwise, you'll be taken away from it.

2.6.2 Bifurcation

Now, we'll consider *families* of autonomous DEs, rather than a single DE, at a time. Here is a family of DEs parametrized by a number r (8A-1 from the homework packet):

$$P' = -P^3 + 12P^2 - 36P + r$$

We want to figure out how the number and stability of critical points depends on the parameter r. Our critical points are defined as P values that cause P' to equal 0, so they are given by $r = -P^3 + 12P^2 - 36P$. If we plot r vs. P, we now have what is called a bifurcation diagram (one that I brutally ripped out the homework packet):



Bifurcation

This diagram has r on the horizontal axis, and the critical points of the DE corresponding to the current r on the vertical axis. Thus, we can see that when r is 0, there is a stable critical point at $P = 0$, and a semistable critical point at $P = 6$. There are also other critical points for higher r values.

Note that the curve divides the plan into subsections. In any section, the direction of the arrows is always the same. This is true for any bifurcation diagram.

2.7 Differential Operators

2.7.1 Definition

Suppose we have a polynomial p. We call p "applied" to the derivative operator D, $p(D)$, a differential operator. Here's an example:

$$p(x) = x^2 + 2x + 1$$
$$p(D) = D^2 + 2D + 1$$
$$p(D)(e^{\alpha x}) = \alpha^2 e^{\alpha x} + 2\alpha e^{\alpha x} + 1$$

These operators $p(D)$ have some nice properties (which all basically arise from the linearity of D). For sufficiently differentiable functions u:

$$(p(D) + q(D))u = p(D)u + q(D)u$$
$$p(D)(c_1u_1 + c_2u_2) = p(D)c_1u_1 + p(D)c_2u_2 \qquad \text{Linearity}$$
$$(g(D)h(D))u = g(D)(h(D))u \qquad \text{Multiplication}$$
$$p(D)e^{\alpha x} = p(\alpha)e^{\alpha x} \qquad \text{Substitution}$$
$$p(D)e^{\alpha x}u = e^{\alpha x}p(D + \alpha)u \qquad \text{Exponential Shift}$$

The last identity, called the Exponential-Shift Rule, can be shown by induction on $D, D^1, D^2, \ldots, D^n$, and the general result follows from linearity of D.

2.7.2 Solving Homogenous Linear DEs with Constant Coefficients

Suppose our equation takes the form:

$$p(D)x = 0$$

Guessing $y = e^{\alpha x}$ and applying the Substitution Rule quickly yields a general solution:

$$p(D)e^{\alpha x} \iff p(a)e^{\alpha x} = 0 \iff p(a) = 0$$

So we can see that roots of the characteristic equation directly yield solutions. Suppose we have a repeated root α, with multiplicity k. Then we can write our characteristic equation as:

$$p(x) = r(x)(x - \alpha)^k$$

with $r(x)$ denoting the "rest" of the characteristic equation. Now, we can easily see that any $x^i e^{\alpha x}$ is a solution, for $i < k$:

$$\begin{aligned} p(D)x^i e^{\alpha x} &= r(D)(D - \alpha)^k x^i e^{\alpha x} \\ &= r(D)e^{\alpha x} D^k x^i && \text{by the Exponential Shift Rule} \\ &= 0 && \text{since } D^k x^i \text{ is 0 if } i < k \end{aligned}$$

2.7.3 Solving Inhomogenous Linear DEs with Constant Coefficients

Suppose p has degree n and our DE takes the form:

$$p(D)y = f(x) \tag{3}$$

Based on $f(x)$, we can guess a solution then try to solve algebraically for particular coefficients. We guess a linear combination of $f(x)$ and its first n derivatives. For example, if $f(x) = 3x + 2$, we would guess the function $Ax + B$. If $f(x) = xe^x$ and $n > 1$, we would guess $Axe^x + Be^x$ because the derivative of xe^x has an e^x term in it.

One small hiccup is that our trial solution satisfies the complementary equation of (3), it will of course never solve (3) since we'll just get 0. The fix is to multiply the trial solution by x^s where s is the smallest integer so that no term in our new trial solution is a multiple of a term in the solution to the complementary equation of (3).

2.7.4 An Important Case: $e^{\alpha x}$

An important case of inhomogenous linear DE is when $f(x) = e^{\alpha x}$.

$$p(D)y = e^{\alpha x}$$

Then this equation has particular solution y_p:

$$y_p = \frac{e^{\alpha x}}{p(a)},\ p(a) \neq 0$$

If $p(a)$ is equal to 0, with multiplicity s, then,

$$y_p = \frac{x^s e^{\alpha x}}{p^{(s)}(a)}$$

Note that the first equation is a special case of the second, with $s = 0$.

2.8 Gain and Resonance

Let's consider a driven spring-block-dashpot system modelled by the usual DE; suppose the driving force is $F_0 \cos \omega t$:

$$mx'' + bx' + kx = F_0 \cos \omega t \tag{4}$$

We find a particular solution using the operator method on the equation $mx'' + bx' + kx = F_0 e^{i\omega t}$ and taking the real part [1] (assuming ω is not a root of the characteristic equation of (4)):

$$\begin{aligned}
\tilde{x}_p &= \frac{F_0 e^{i\omega t}}{m(i\omega)^2 + b(i\omega) + k} \\
&= \frac{F_0(\cos(\omega t) + i\sin(\omega t))}{-m\omega^2 + ib\omega + k} \\
&= \frac{F_0(\cos(\omega t) + i\sin(\omega t))}{-m\omega^2 + ib\omega + k} \cdot \frac{-m\omega^2 - ib\omega + k}{-m\omega^2 - ib\omega + k} \\
&= \frac{F_0((k - m\omega^2)\cos(\omega t) + b\omega\sin(\omega t)) - iF_0(b\omega\cos(\omega t) + (k - m\omega^2)\sin(\omega t))}{(k - m\omega^2)^2 + b^2\omega^2}
\end{aligned}$$

Now, taking the real part, we have:

$$x_p = \operatorname{Re} \tilde{x}_p = \frac{F_0((k - m\omega^2)\cos(\omega t) + b\omega\sin(\omega t))}{(k - m\omega^2)^2 + b^2\omega^2} \tag{5}$$

Let's define $\phi = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right)$. Thus, $\cos\phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$ and $\sin\phi = \frac{b\omega}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$. Substituting these into (5), we have:

$$\frac{F_0\sqrt{(k - m\omega^2)^2 + b^2\omega^2}(\cos\phi\cos(\omega t) + \sin\phi\sin(\omega t))}{(k - m\omega^2)^2 + b^2\omega^2}$$

Applying $\cos\alpha + \beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we finally arrive at the solution to our original equation (yay!):

$$x_p = \frac{F_0\cos(\omega t - \phi)}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

[1] If $f(x)$ where to take the form $F_0 \sin \omega t$, we would instead take the imaginary part

The **amplitude gain** of the system is defined as the ratio of the output amplitude to the input amplitude:

$$\frac{F_0}{\sqrt{(k-m\omega^2)^2+b^2\omega^2}} \Big/ F_0 = \frac{1}{\sqrt{(k-m\omega^2)^2+b^2\omega^2}}$$

The **phase lag** of the system is just $\phi = \tan^{-1}\left(\frac{b\omega}{k-m\omega^2}\right)$.

Notice that if we define $p(x) = mx^2 + bx + k$, the **complex gain** is defined as $\frac{1}{p(i\omega)}$. Then amplitude gain is just the norm of this complex number, and the phase lag is just the argument.

2.9 Phase Portraits

The signs/complexity of the eigenvalues of a 2-by-2 system of linear DEs give us a general picture of of trajectories the system can take. Since $\det A = \lambda_1\lambda_2$ and $\operatorname{tr} A = \lambda_1 + \lambda_2$, the characteristic polynomial can be written as:

$$x^2 - (\operatorname{tr} A)x + \det A = 0$$

From the quadratic formula, we have:

$$x = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}$$

Thus, we have complex roots if $\frac{(\operatorname{tr} A)^2}{4} < \det A$. In other words, if we graph the $\operatorname{tr} A$ versus $\det A$, matrices above the parabola $y = \frac{(\operatorname{tr} A)^2}{4}$ will have complex roots. Trajectories in the northwest quadrant, which have negative trace and positive determinant (i.e, both eigenvalues have negative real part), are called stable, as all trajectories eventually end at 0.

Let's go through all the trace-determinant combinations, with their associate eigenvalues:

complex oscillation

- **real part zero** $\det A = 0$, $\det A > \frac{(\operatorname{tr} A)^2}{4}$: ellipses
- **real part positive** $\det A > 0$, $\det A > \frac{(\operatorname{tr} A)^2}{4}$: unstable spirals
- **real part negative** $\det A < 0$, $\det A > \frac{(\operatorname{tr} A)^2}{4}$: stable spirals

2 real eigenvalues no oscillation

- **one positive, one negative** $\det A < 0$: hyperbolas, ending parallel to larger eigenvalue
- **two positive** $\operatorname{tr} A > 0$, $\det A < \frac{(\operatorname{tr} A)^2}{4}$: unstable parabolas that extend parallel to larger eigenvalue's eigenvector

two negative $\operatorname{tr} A < 0$, $\det A < \frac{(\operatorname{tr} A)^2}{4}$: stable parabolas that go to zero parallel to larger eigenvalue's eigenvector

two repeated positive $\operatorname{tr} A > 0$, $\det A = \frac{(\operatorname{tr} A)^2}{4}$: unstable star, any direction is possible as a linear combination of the two eigenvectors, and both components grow at same rate

two repeated negative $\operatorname{tr} A < 0$, $\det A = \frac{(\operatorname{tr} A)^2}{4}$: stable star, any direction is possible as a linear combination of the two eigenvectors, and both components shrink at same rate

one positive, one zero $\operatorname{tr} A > 0$, $\det A = 0$: unstable critical line (comb)

one negative, one zero $\operatorname{tr} A < 0$, $\det A = 0$: stable critical line (comb)

2.10 Nonlinear Systems

To analyze an autonomous nonlinear system:

$$x' = f(x, y)$$
$$y' = g(x, y)$$

at a point (x_0, y_0), we can approximate it by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = J_{(x_0, y_0)} \begin{bmatrix} x \\ y \end{bmatrix}$$

Where J is the Jacobian matrix:

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

evaluated at (x_0, y_0). Note that for a single equation $u' = f(x)$ we can apply the same method. We just get $x' = f'(x)x$, since the Jacobian has just one element.

If there are pure complex eigenvalues, zero eigenvalues, or repeated eigenvalues (basically if anything is weird), we can't use this method to approximate the system because it is not **structurally stable**.

2.10.1 Structural Stability

A system is structurally stable if small changes to it's parameters don't change the geometry or stability of its critical points (tweaking coefficients doesn't make drastic changes). Only (regular) spirals, saddles, and nodes are stable.

2.10.2 Population Dynamics

Let x be the population of some prey species, and y the population of some predator species. Assuming no predators, the prey grows naturally with a fixed growth rate:

$$x' = ax$$

Conversely, without prey, the predators decline at a fixed loss rate:

$$y' = -by$$

In each other's presence, we assume the two species interact with rate xy, with each encounter harming the prey by p, and benefitting the predators by q. Hence, we have:

$$\begin{aligned} x' &= ax - pxy \\ y' &= -by + qxy \end{aligned}$$

Writing these two equations like so:

$$\begin{aligned} x' &= x(a - py) \\ y' &= y(-b + qx) \end{aligned}$$

it is easy to see that there are two critical points: $(0,0)$ (mutual extinction), and $\left(\frac{b}{q}, \frac{a}{p}\right)$.

2.11 Fourier Series

The Fourier coefficients of a 2π-periodic function f are:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \end{aligned}$$

f's full Fourier series is:

$$\frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nt + \sum_{1}^{\infty} b_n \sin nt$$

2.11.1 Arbitrary Period

Suppose we have a piecewise continuous function $f(t)$ with period $P = 2L$. Then if we "stretch" the function by $\frac{P}{2\pi}$, we'll get a version of f with period 2π. More concretely, let's define the function:

$$g(t) = f\left(\frac{P}{2\pi}t\right) = f\left(\frac{2L}{2\pi}t\right) = f\left(\frac{L}{\pi}t\right)$$

A quick check that g has period 2π:

$$g(t+2\pi) = f\left(\frac{L}{\pi}(t+2\pi)\right) = f\left(\frac{L}{\pi}t + \left(\frac{2L\pi}{\pi}\right)\right) = f\left(\frac{L}{\pi}t + 2L\right) = f\left(\frac{L}{\pi}t\right) = g(t)$$

As usual, the Fourier coefficients of g are:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\, dt$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\, dt$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin nt\, dt$$

We can now retrieve the Fourier coefficients for $f(t)$ using the substitution $t = \frac{\pi t'}{L}$, which means:

$$dt = \frac{\pi}{L}dt'$$
$$g\left(\frac{\pi t'}{L}\right) = f\left(\frac{L}{\pi}\frac{\pi t'}{L}\right) = f\left(t'\right)$$

Hence:

$$a_0 = \frac{1}{\pi}\int_{-\pi\frac{L}{\pi}}^{\pi\frac{L}{\pi}} g\left(\frac{\pi}{L}t'\right)\frac{\pi}{L}dt' = \frac{1}{L}\int_{-L}^{L} f(t')\, dt'$$

Likewise:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t')\cos\frac{n\pi t'}{L}\, dt'$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t')\sin\frac{n\pi t'}{L}\, dt'$$

2.11.2 Even/Odd Extensions

Recall that if f is even, then:

$$\int_{-L}^{L} f(t)\, dt = 2\int_{0}^{L} f(t)\, dt$$

Conversely, if f is odd

$$\int_{-L}^{L} f(t)\, dt = 0$$

Also, recall that the produce of two odd or even function is even, while the product of an even and odd function is odd. These properties will be used to calculate Fourier coefficients for an extended function f. Suppose f is defined over $(0, L)$. To make it periodic with period 2L, two common ways to extend f to $(-L, 0)$ are to use $f(-t) = f(t)$ (even extension) or $f(-t) = -f(t)$ (odd extension).

The even extension has Fourier cosine series:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{2}{L}\int_{0}^{L} f(t)\,dt \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{\pi n t}{L}\,dt = \frac{2}{L}\int_{0}^{L} f(t)\cos\frac{\pi n t}{L}\,dt \\ b_n &= 0 \end{aligned}$$

The odd extension has Fourier sine series:

$$\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{\pi n t}{L}\,dt = \frac{2}{L}\int_{0}^{L} f(t)\sin\frac{\pi n t}{L}\,dt \end{aligned}$$

2.11.3 Using Fourier Series to Solve DEs

Suppose f is even and we have a DE of the form:

$$x'' + kx = f(t) \tag{6}$$

Let's look for a periodic solution $x = \sum_0^\infty A_n \cos nt$. Plugging this and the cosine series for f in, we have:

$$-n^2 \sum_0^\infty A_n \cos nt + k \sum_0^\infty A_n \cos nt = \sum_0^\infty a_n \cos nt$$

Matching like terms with like terms, we get:

$$A_n = \frac{a_n}{k - n^2}$$

which gives us the Fourier coefficients of the solution to (6).

Note that if k is positive (perhaps we might suggestively then write $k = \omega^2$), the solutions the complementary equation of (6) take the form $A\cos\omega t + B\sin\omega t$. If there is a term in the Fourier series solution with n close to ω, this is called **near resonance**. Looking at this particular A_n, we can see that $k - n^2$ will be very small, so the term will have a large magnitude, which makes intuitive sense.

If $f(t)$ is odd, we can basically do the same thing, except with a sine series for $f(t)$.

2.12 The Heat Equation

Suppose we have a bar of length L. The Heat Equation for the bar is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with initial conditions:

$$u(x,0) = u_0(x)$$

and boundary conditions:

$$u(0,t) = T_0,\ u(L,t) = T_L$$

One way to solve the equation is to look for solutions of the form $u(x,t) = v(x)w(t)$, also known as separation of variables. Plugging this trial solution in, we get

$$\dot{w}(t)v(x) = kv''(x)w(t)$$
$$\frac{\dot{w}(t)}{w(t)} = k\frac{v''(x)}{v(x)}$$

Let's call the quantity on either side of the last equation $c(x,t)$. Notice that $\frac{\partial c}{\partial x} = \frac{\partial}{\partial x}\frac{\dot{w}(t)}{w(t)} = 0$ and $\frac{\partial c}{\partial t} = \frac{\partial}{\partial t}k\frac{v''(x)}{v(x)} = 0$, so $c(x,t)$ must just be a constant, which I'll henceforth call c. Thus, we now have:

$$c = k\frac{v''(x)}{v(x)}$$

Plugging in a guess $v_n(x) = \sin nx$ [2], we get:

$$c = k\frac{-n^2 \sin nx}{\sin nx} = -kn^2$$

Now, we can solve for a corresponding $w_n(t)$ by using our c value:

$$\frac{\dot{w}(t)}{w(t)} = -kn^2 \implies \dot{w}(t) = -kn^2 w(t) \implies w(t) = e^{-kn^2 t}$$

Thus we have found a family of solutions to the Heat Equation: $u_n(x,t) = e^{-kn^2 t}\sin nx$ ($u_n(x,t) = e^{-kn^2 t}\cos nx$ also works). Using superposition and the Fourier coefficients of $u_0(x)$, a_0, a_n, b_n, we get the full solution to the Heat Equation:

$$u(x,t) = \frac{a_0}{2} + \sum_1^\infty a_n e^{-kn^2 t}\cos nx + \sum_1^\infty b_n e^{-kn^2 t}\sin nx$$

[2] Your "guess" should obey the boundary conditions; sometimes this might mean using $v_n(x) = \cos nx$

And we can see easily that this solution obeys the boundary conditions:

$$\begin{aligned}u_0(x) = u(x,0) &= \frac{a_0}{2} + \sum_1^\infty a_n e^{-kn^2(0)} \cos nx + \sum_1^\infty b_n e^{-kn^2(0)} \sin nx \\ &= \frac{a_0}{2} + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx\end{aligned}$$

3 Supplementary Content

3.1 Euler's Method

3.1.1 Classic Basic Method

Suppose we have an DE that is not exactly solvable of the form:

$$y' = f(x,y)$$

Using step size h and starting at (x_0, y_0), we generate a new approximated point in the solution using the following rule:

$$\begin{aligned}x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h \cdot f(x_n, y_n)\end{aligned}$$

3.1.2 Improved Euler's Method

We now use the update rule:

$$\begin{aligned}k_1 &= f(x_n, y_n) \\ u_{n+1} &= y_n + hk_1 \\ k_2 &= f(x_{n+1}, u_{n+1}) \\ y_{n+1} &= y_n + \frac{1}{2}h(k_1 + k_2)\end{aligned}$$

In other words, we use the average slope on the interval $[x_n, x_{n+1}]$ to create our new y_{n+1}

3.2 Existence and Uniqueness For Linear DEs

Suppose we have a linear DE of the form:

$$y'' + p(x)y' + q(x)y = f(x)$$

If p, q, and f are continuous an open interval I, then for any $a \in I$, b_1, b_2 such that $y(a) = b_1$ and $y'(a) = b_2$, , there exists a unique solution over the *entire* interval.

3.3 Wronskian

The Wronskian of two functions f and g is defined as the determinant:

$$\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Note that if f and g are linearly dependent (constant multiples of each other), the Wronskian is 0 everywhere. Furthermore, if our equation takes the familiar form:

$$y'' + p(x)y' + q(x)y = f(x)$$

Then two solutions y_1 and y_2 on an interval I have Wronskian 0 everywhere if linearly dependent, and nonzero Wronskian everywhere if linearly independent.

3.4 Damped Simple Harmonic Motion

Starting with the standard equation for SHM:

$$mx'' + cx' + kx = 0$$

Rewrite with $p = \frac{c}{2m} > 0$ and $\omega_0 = \sqrt{\frac{k}{m}}$ (the undamped frequency):

$$x'' + 2px' + \omega_0{}^2 x = 0$$

Then the characteristic equation has roots:

$$r1, r2 = -p \pm \sqrt{p^2 - \omega_0{}^2}$$

whose real/complex-ness depend on the sign of:

$$p^2 - \omega_0{}^2 = \frac{c^2 - 4km}{4m^2}$$

We call $c_{cr} = \sqrt{4km}$ the critical damping, and have three cases based on its relation with the actual damping factor c:

$$\begin{cases} x(t) = c_1 e^{r_1 t} + c_1 e^{r_2 t}, & \text{if } c_{cr} > c \text{ (overdamping)} \\ x(t) = (c_1 + c_2 x)e^{-pt}, & \text{if } c_{cr} = c \text{ (critical damping)} \\ x(t) = e^{-pt}\left(A\cos\omega_1 t + B\sin\omega_1 t\right) \text{ where } \omega_1 = i\sqrt{\omega_0^2 - p^2}, & \text{if } c_{cr} < c \text{ (underdamping)} \end{cases}$$

This condition due to c_{cr} is essentially just looking at the discriminant of the characteristic equation. The normal intuition applies. For example, if the discriminant is negative, we have complex roots and we would expect oscillation.

3.5 Systems of Linear DEs

Suppose we have a high-order linear differential equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots a_1 y^{(1)} + a_0 y = f(x)$$

We can introduce new variables

$$\begin{aligned} y_1 &= y' \\ y_2 &= y_1' \\ &\dots \\ y_{n-1} &= y_{n-2}' \end{aligned}$$

to make this a system of first order differential equations:

$$a_n y_{n-1}' + a_{n-1} y_{n-1} + \dots a_1 y_1 + a_0 y = f(x)$$

Now, we have $n-1$ equations, and $n-1$ variables, which we can solve using a method like elimination (or numerically in practice).

3.5.1 Elimination

Elimination generally involves writing one equation in terms of another, producing an equation of higher degree but with fewer variables. For example, consider the system (4B-3 in the IAP 2025 homework packet):

$$\begin{aligned} x' &= x + y \\ y' &= 4x + y \end{aligned}$$

Notice that we can rewrite $y = x' - x$ and $y' = x'' - x$. Substituting these into the second equation, we get:

$$\begin{aligned} x'' - x' &= 4x + (x' - x) \\ x'' - 2x' - 3x &= 0 \end{aligned}$$

which we can solve via the usual means.

3.5.2 Decoupling

Suppose we have a linear DE with constant coefficients of the form:

$$\mathbf{x}' = A\mathbf{x}$$

and A is diagonalizable. This means write the system a little more simply. Applying the diagonalization:

$$\begin{aligned} \mathbf{x}' &= S\Lambda S^{-1}\mathbf{x} \\ S^{-1}\mathbf{x}' &= S^{-1}S\Lambda S^{-1}\mathbf{x} \\ S^{-1}\mathbf{x}' &= \Lambda S^{-1}\mathbf{x} \\ (S^{-1}\mathbf{x})' &= \Lambda(S^{-1}\mathbf{x}) \end{aligned}$$

Let's call $S^{-1}\mathbf{x}$, the representation of $\mathbf{x}$ in the eigenbasis, $\mathbf{y}$. Then we can say:

$$\mathbf{y}' = \Lambda\mathbf{y}$$

3.6 Another way of computing e^{At}

Recall from 18.06 that $e^{At}\mathbf{x}(0)$ is the solution to the system of DEs:

$$\mathbf{x}' = A\mathbf{x}$$

We can also write the solution as some linear combination of the eigenfunctions (the first way we solved this equation). Suppose we have eigenbasis $v_1, \ldots, v_n$ with $\lambda_1, \ldots, \lambda_n$. Then the general solution is $c_1 v_1 e^{\lambda_1 t}, \ldots, c_n v_n e^{\lambda_n t}$. But where do these c_i's come from? The initial conditions of course. As a tool, we'll use what's called a Fundamental Matrix of our system:

$$\Phi(t) = \begin{bmatrix} v_1 e^{\lambda_1 t} & \cdots & v_n e^{\lambda_n t} \end{bmatrix}$$

Note that since each column of this matrix is a solution to $\mathbf{x}' = A\mathbf{x}$, the entire matrix itself also obeys it, i.e. $\Phi'(t) = A\Phi(t)$. Now, if we put the c_is in a column vector $\mathbf{c}$, we can more tersely write the general solution as:

$$\Phi(t)\mathbf{c}$$

The utility of this comes when we set $t = 0$:

$$\begin{aligned} \Phi(0)\mathbf{c} &= \mathbf{x}(0) \\ \mathbf{c} &= \Phi(0)^{-1}\mathbf{x}(0) \end{aligned}$$

All we're doing here is extracting the actual coefficients for writing $\mathbf{x}(0)$ using the eigenbasis; think of $\Phi(t)^{-1}$ like a change of basis matrix. This allows us to write $\Phi(t)\mathbf{c}$ as $\Phi(t)\Phi(0)^{-1}\mathbf{x}(0)$. Since this is just another way of writing the general solution to our equation, we have

$$\begin{aligned} e^{At}\mathbf{x}(0) &= \Phi(t)\Phi(0)^{-1}\mathbf{x}(0) \\ e^{At} &= \Phi(t)\Phi(0)^{-1} \end{aligned}$$

which gives us another way of computing e^{At}.

3.7 Inhomogeneous Linear Systems (Exponential Response Formula)

Let's start again with a linear system, but this time it's not homogenous:

$$\mathbf{x}' = A\mathbf{x} + e^{\alpha t}\mathbf{K}$$

where $\mathbf{K}$ is a vector of constants. Like before, we'll guess a particular solution $e^{\alpha t}\mathbf{v}$. Plugging this in, we get:

$$\begin{aligned}\alpha e^{\alpha t}\mathbf{v} &= Ae^{\alpha t}\mathbf{v} + e^{\alpha t}\mathbf{K} \\ \alpha\mathbf{v} &= A\mathbf{v} + \mathbf{K} \\ \alpha\mathbf{v} - A\mathbf{v} &= \mathbf{K} \\ (\alpha I - A)\mathbf{v} &= \mathbf{K} \\ \mathbf{v} &= (\alpha I - A)^{-1}\mathbf{K}\end{aligned}$$

Once again, if the inhomogeneous term has the form $\cos\alpha t$ or $\sin\alpha t$, we can replace it with $e^{\alpha i t}$ and then take the real/complex part of the solution we get.

3.7.1 Variation of Parameters

More generally, suppose we have a problem:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{q}(t) \tag{7}$$

The general solution to the homogenous equation is $\Phi(t)\mathbf{c}$. Let's replace c with a vector of functions $\mathbf{u}(t)$. In other words, lets let our c_is from the previous part be functions instead, which we can solve for. Therefore, our new solution is going to take the form $\mathbf{x} = \Phi(t)\mathbf{u}(t)$. Plugging this into (7) we have:

$$\begin{aligned}(\Phi(t)\mathbf{u}(t))' &= A(\Phi(t)\mathbf{u}(t)) + \mathbf{q}(t) \\ A\Phi(t)\mathbf{u}(t) + \Phi(t)\mathbf{u}'(t) &= A\Phi(t)\mathbf{u}(t) + \mathbf{q}(t)\end{aligned}$$

Cancelling the $A\Phi(t)\mathbf{u}(t)$s from each side gives:

$$\begin{aligned}\Phi(t)\mathbf{u}'(t) &= \mathbf{q}(t) \\ \mathbf{u}'(t) &= \Phi(t)^{-1}\mathbf{q}(t) \\ \mathbf{u}(t) &= \int \Phi(t)^{-1}\mathbf{q}(t)\end{aligned}$$

Thus, $\mathbf{x} = \Phi(t)\int\Phi(t)^{-1}\mathbf{q}(t)$.

3.8 More Fourier Series

3.8.1 Convergence

A few definitions first:

piecewise continuous f is piecewise continuous over an interval I if it has finite jump discontinuities over I, and the left and right-sided limits at each discontinuity exist and are finite (this rules out something like tan, where the limits are not finite).

piecewise smooth f is piecewise smooth if f' is piecewise continuous.

The Fourier series for f is guaranteed to converge over interval I if f is piecewise smooth over I. At a discontinuity t_0, the Fourier series takes the value $\frac{f(t_0-)+f(t_0+)}{2}$, i.e. the mean of the values the function takes when approaching the discontinuity from either side.

3.8.2 Termwise Differentiation

If f is continuous and f' is *piecewise smooth* over I, then the series obtain by termwise differentiation of f's Fourier series converges to f' on I.