18.901, FALL 2024 — FINAL EXAM

Each part of each problem is worth 10 points. Your best 6 scores out of the 8 total parts will be counted, so the exam is graded out of 60 points.

Problem 1. Let X be a topological space, let $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$ be a sequence of closed, connected subspaces of X, and let $Z := \bigcap_{n \in \mathbb{N}} Z_n$.

(a) Show that, if X is compact Hausdorff, then Z is connected.

(b) For $X = \mathbb{R}^2$, give an example to show that Z need not be connected.

Problem 2. Let $f : \mathrm{S}^1 \to \mathrm{S}^1$ be a continuous map such that $f(-z) = f(z)$ for all $z \in \mathrm{S}^1$. Show that $\deg(f)$ is even.

Problem 3. Let $T := \mathrm{S}^1 \times \mathrm{S}^1$ be the torus.

(a) Give an example of a universal covering $p : Y \to T$ (and justify that it is one).

(b) Let $t_0 \in T$, and let $f : T \to T$ be a continuous map such that $f(t_0) = t_0$ and such that the induced map $f_* : \pi_1(T, t_0) \to \pi_1(T, t_0)$ is equal to the identity map on $\pi_1(T, t_0)$. Show that f is homotopic to the identity map on T. Hint: Use the previous part.

Problem 4. Show that S^2 is not homeomorphic to S^3.

Problem 5. In this problem, let $D \subseteq \mathbb{C}$ be the closed disk $\{z \in \mathbb{C} : |z| \leq 1\}$ and let $D' \subseteq \mathbb{C}$ be the open disk $\{z \in \mathbb{C} : |z| < 1\}$.

(a) Let X be a path connected and locally path connected topological space, and let $p : D \to X$ be a covering map. Show that p is a homeomorphism.

(b) Show that the statement in the previous part does not remain true when D is replaced by D'.