18.901, FALL 2024 — HOMEWORK 0

Each part of each main problem is worth 5 points, making this homework total to 15 points. Each part of the bonus problem is worth 0.5 additional points.

Main problems

Problem 1. Show that there is an injection $\mathbb{R} \to (0,1)$.

Problem 2. Let X and Y be sets, and let $f : X \to Y$ and $g : Y \to X$ be injective functions. The Cantor–Bernstein–Schröder theorem asserts that, given this situation, there exists a bijection between X and Y. In this problem, you will prove this theorem.

Set $X_0 := X$ and $Y_0 := Y$, and for n a positive integer, recursively define $X_n := g(f(X_{n-1}))$ and $Y_n := f(g(Y_{n-1}))$.

(a) Show that, for each $n \geq 0$, we have containments

$$X_n \supseteq g(Y_n) \supseteq X_{n+1}, \qquad Y_n \supseteq f(X_n) \supseteq Y_{n+1}.$$

(b) Let $F : X \to Y$ be the function defined by

$$F(x) := \begin{cases} f(x) & \text{if } x \in X_n \smallsetminus g(Y_n) \text{ for some } n \geq 0 \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

(note that the condition for the first case failing to hold for $n = 0$ means that $x \in g(Y)$, and since g is injective, there then exists a unique $y \in Y$ such that $g(y) = x$; we denote this y by $g^{-1}(x)$ in the second case). Show that F is bijective.

Bonus problems

Problem 3. For a set X, we let $\mathcal{P}(X)$ denote the powerset of X, i.e. the set of all subsets of X.

(a) Show that there is a bijection between $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$.

Hint: You may want to use Problems 1 and 2. Moreover, you may use without proof the following facts:

- for any sequence $\{a_n\}_{n\in\mathbb{N}}$ where $a_n \in \{0,1\}$ for each $n \in \mathbb{N}$, the series $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$ converges;
- for any $x \in (0,1)$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ where $a_n \in \{0,1\}$ for each $n \in \mathbb{N}$ such that $x = \sum_{n=0}^{\infty} \frac{a_n}{2^n}$.

(b) For any set X, show that there is a bijection between $\mathcal{P}(X) \times \mathcal{P}(X)$ and $\mathcal{P}(X \sqcup X)$.

(c) Show that there is a bijection between $\mathbb{N} \sqcup \mathbb{N}$ and $\mathbb{N}$.

(d) Deduce from the previous parts of the problem that there exists a bijection between $\mathbb{R}$ and $\mathbb{R}^2$.