18.901, FALL 2024 — HOMEWORK 1

Each part of each main problem is worth 5 points, so this homework is graded out of 50 points. Each part of each bonus problem is worth 0.5 additional points.

Main problems

Problem 1. Equip $\mathbb{R}$ with its standard metric. Let $r : \mathbb{R} \to \mathbb{R}$ be the function defined by $r(x) := -x$ and, for $a \in \mathbb{R}$, let $t_a : \mathbb{R} \to \mathbb{R}$ be the function defined by $t_a(x) := x + a$.

(a) Show that r and t_a (for any $a \in \mathbb{R}$) are isometries.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be an isometry. Show that there exists $a \in \mathbb{R}$ such that $f = t_a$ or $f = t_a \circ r$.

Problem 2. Let X and $Y_1, \ldots, Y_n$ be metric spaces, and equip the product $\prod_{i=1}^n Y_i$ with the ℓ^∞ product metric.

(a) Let $f : X \to \prod_{i=1}^n Y_i$ be a function, and for each $1 \le i \le n$, let $f_i : X \to Y_i$ denote the i-th component function of f, i.e. the composition of f with the projection function $\prod_{i=1}^n Y_i \to Y_i$ sending $(y_1, \ldots, y_n) \mapsto y_i$. Show that f is continuous if and only if all of the functions $f_1, \ldots, f_n$ are continuous.

(b) We now specialize to the case that $X = Y_1 = \cdots = Y_n$. Show that the diagonal map $\delta : X \to X^n$, sending $x \mapsto (x, \ldots, x)$, is continuous.

Problem 3. Let X be a metric space, let Y be a subset of X, and let $\overline{Y}$ be the closure of Y in X. Show that $\overline{Y}$ is closed in X.

Problem 4. Let X be a metric space. Let $\mathcal{P}_{\mathrm{cb}}(X)$ denote the set of closed and bounded subsets of X. For $A, B \in \mathcal{P}_{\mathrm{cb}}(X)$, define

$$d_{\mathrm{H}}^{\mathrm{pre}}(A,B) := \sup_{b \in B} \inf_{a \in A} d_X(a,b), \qquad d_{\mathrm{H}}(A,B) := \max(d_{\mathrm{H}}^{\mathrm{pre}}(A,B), d_{\mathrm{H}}^{\mathrm{pre}}(B,A)).$$

(a) Prove that d_{H} is a metric on $\mathcal{P}_{\mathrm{cb}}(X)$. This is called the *Hausdorff metric*.

(b) Prove that the diameter function $\operatorname{diam} : \mathcal{P}_{\mathrm{cb}}(X) \to \mathbb{R}$ is continuous (where the domain is equipped with the Hausdorff metric and the codomain with the standard metric).

(c) We now specialize to the case that X is $\mathbb{R}^2$ with its standard metric. Let $C \subset \mathbb{R}^2$ be a circle of radius 1, and for each integer $n \ge 3$, let P_n be a regular n-gon inscribed in C. Show that the sequence $\{P_n\}_{n \ge 3}$ converges to C in the metric space $\mathcal{P}_{\mathrm{cb}}(\mathbb{R}^2)$.

Problem 5. Let p be a prime number, equip $\mathbb{Q}$ with the p-adic metric, and equip $\mathbb{Q} \times \mathbb{Q}$ with the ℓ^∞ product metric (each factor still being equipped with the p-adic metric).

(a) Show that addition and multiplication define continuous maps $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$.

(b) Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function defined by a polynomial with rational coefficients in one variable. Prove that f is continuous.

Problem 6. Let X and Y be sets and let $f : X \to Y$ be a function. Let Z be a set, and let Z^X and Z^Y denote the sets of functions $X \to Z$ and $Y \to Z$, respectively. Let $\rho_f : Z^Y \to Z^X$ be the function sending $g \mapsto g \circ f$. Show that if f is surjective, then ρ_f is injective, and that if f is bijective, then ρ_f is bijective. (We assume the axiom of choice in this class.)

Problem 7. Let X be a set. Recall that an *equivalence relation on* X is a subset $E \subseteq X \times X$ satisfying the following properties:

- *Reflexivity*: For $x \in X$, we have $(x, x) \in E$.
- *Symmetry*: For $x, y \in X$, if $(x, y) \in E$ then $(y, x) \in E$.
- *Transitivity*: For $x, y, z \in X$, if $(x, y) \in E$ and $(y, z) \in E$, then $(x, z) \in E$.

Let us fix an equivalence relation E on X for this problem, and define an *E–equivalence class* to be a nonempty subset $C \subseteq X$ such that for $x, y \in X$, if $x \in C$, then $y \in C$ if and only if $(x, y) \in E$. Let X/E denote the subset of the powerset $\mathcal{P}(X)$ consisting of the E–equivalence classes.

(a) Show that for each $x \in X$, there exists a unique E–equivalence class containing x.

We call the E–equivalence class containing $x \in X$ the *E–equivalence class of* x and denote it by $q_E(x)$. This defines a function $q_E : X \to X/E$, which we refer to as the *quotient function* associated to E.

(b) Let Z be a set, and let $\rho_{q_E} : Z^{X/E} \to Z^X$ be as in Problem 6. Show that ρ_{q_E} is injective, and moreover that a function $f : X \to Z$ is in the image of ρ_{q_E} if and only if $f(x) = f(y)$ for all $(x, y) \in E$.

(c) Let $E' \subseteq (X \times X) \times (X \times X)$ be the subset consisting of those elements $((x, y), (x', y'))$ such that $(x, x') \in E$ and $(y, y') \in E$. Show that E' is an equivalence relation on $X \times X$, and that there exists a unique bijection $u : (X \times X)/E' \to X/E \times X/E$ such that the composition

$$X \times X \xrightarrow{q_{E'}} (X \times X)/E' \xrightarrow{u} X/E \times X/E$$

is equal to the function $q_E \times q_E : X \times X \to X/E \times X/E$.

Problem 8. This problem will make reference to Problem 7. Let X be a metric space. Let $\widetilde{X}$ be the set of Cauchy sequences in X. Let E be the subset of $\widetilde{X} \times \widetilde{X}$ consisting of those pairs of sequences $(\{x_n\}_{n\in\mathbb{N}}, \{x'_n\}_{n\in\mathbb{N}})$ such that for all $\epsilon > 0$, there exists $N > 0$ such that $d(x_n, x'_n) < \epsilon$ for all $n > N$.

(a) Show that E is an equivalence relation on $\widetilde{X}$.

Let $\widehat{X}$ denote the set of E-equivalence classes $\widetilde{X}/E$.

(b) Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two Cauchy sequences in X. Show that $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

Using that $\mathbb{R}$ is complete, it follows from (b) that we may define a function $\widetilde{d} : \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ by the formula

$$\widetilde{d}(\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}) := \lim_{n\to\infty} d(x_n, y_n).$$

(c) Show that there is a unique function $\widehat{d} : \widehat{X} \times \widehat{X} \to \mathbb{R}$ such that the composition

$$\widetilde{X} \times \widetilde{X} \xrightarrow{q_E \times q_E} \widehat{X} \times \widehat{X} \xrightarrow{\widehat{d}} \mathbb{R}$$

is equal to $\widetilde{d}$, and show that $\widehat{d}$ is a metric on $\widehat{X}$.

In the next two parts of the problem, we regard $\widehat{X}$ as a metric space via the metric $\widehat{d}$.

(d) Prove that $\widehat{X}$ is complete.

(e) Let $\widetilde{i} : X \to \widetilde{X}$ be the function sending x to the constant sequence $(x, x, x, \ldots)$, and let $i : X \to \widehat{X}$ be the composite $q_E \circ \widetilde{i}$. Prove that i is an isometry and that the image of i is a dense subset of $\widehat{X}$.

(f) Suppose given another metric d' on $\widehat{X}$ with respect to which $i : X \to \widehat{X}$ is also an isometry with dense image. Show that $d' = \widehat{d}$.