18.901, FALL 2024 — HOMEWORK 2

Each part of each main problem is worth 5 points, so this homework is graded out of 50 points. Each part of each bonus problem is worth 0.5 additional points.

Main problems

Problem 1. We say that two metrics d_1 and d_2 on a set X are equivalent if there exist real numbers $c_1 > 0$ and $c_2 > 0$ such that for all $x, y \in X$, we have $d_1(x,y) \le c_1 d_2(x,y)$ and $d_2(x,y) \le c_2 d_1(x,y)$.

(a) Let $X_1, \dots, X_n$ be metric spaces. For any real number $q > 1$, show that the ℓ^∞ product metric and the ℓ^q product metric on $\prod_{i=1}^n X_i$ are equivalent.

(b) Let X be a set and let d_1 and d_2 be two metrics on X. Let $\mathcal{T}_1$ be the topology induced by d_1 and let $\mathcal{T}_2$ be the topology induced by d_2. Show that if d_1 and d_2 are equivalent, then $\mathcal{T}_1 = \mathcal{T}_2$.

Problem 2. Let X be a topological space and let Y be a subset of X. We define:

- the interior of Y (in X) to be the subset Y° of X consisting of those points $x \in X$ such that there exists a neighborhood U of x such that $U \subseteq Y$;
- the closure of Y (in X) to be the subset $\overline{Y}$ of X consisting of those points $x \in X$ such that for every neighborhood U of x, we have that $U \cap Y$ is nonempty.

(a) Show that $Y^\circ \subseteq Y$ and that Y° is open in X. Moreover, show that if U is any open subset of X such that $U \subseteq Y$, then $U \subseteq Y^\circ$.

(b) Show that $Y \subseteq \overline{Y}$ and that $\overline{Y}$ is closed in X. Moreover, show that if Z is any closed subset of X such that $Y \subseteq Z$, then $\overline{Y} \subseteq Z$.

(c) Suppose that the topology on X is induced by a metric d. Show that the the closure $\overline{Y}$ as defined above agrees with the one defined in terms of the metric from Lecture 1.

Problem 3. We say that a topological space X is Hausdorff if for any two distinct points $x, y \in X$, there exist disjoint neighborhoods of x and y, i.e. a neighborhood U of x and a neighborhood V of y such that $U \cap V = \varnothing$.

(a) Let X be a topological space. Show that if X is metrizable, then it is Hausdorff; and show that if X is Hausdorff, then it is T_1.

(b) Let X be an infinite set equipped with the cofinite topology. Show that X is not Hausdorff (and hence not metrizable by (a)).

(c) Let X be a topological space and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X converging to two points $x, x' \in X$. Show that if X is Hausdorff, then $x = x'$.

Problem 4.

(a) Let X be a set, let $\mathcal{B}$ be a basis for a topology on X, and let $\mathcal{T} = \mathcal{T}_\mathcal{B}$ be the topology generated by $\mathcal{B}$. Show that a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges to a point $x \in X$ with respect to the topology $\mathcal{T}$ if and only if for every $B \in \mathcal{B}$ that contains x, there exists $N > 0$ such that $x_n \in B$ for all $n > N$.

(b) Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers and $x \in \mathbb{R}$. Show that $\{x_n\}_{n \in \mathbb{N}}$ converges to x with respect to the lower limit topology on $\mathbb{R}$ if and only if for all $\epsilon > 0$, there exists $N > 0$ such that $x \le x_n < x + \epsilon$ for all $n > N$ (i.e., the sequence converges to x "from the right").

Bonus problems

Problem 5. Let X be a set. Define $\mathcal{T}_{\text{cocount}} \subseteq \mathcal{P}(X)$ to consist of those subsets U of X such that $U = \varnothing$ or $X \setminus U$ is countable.

(a) Show that $\mathcal{T}_{\text{cocount}}$ is a topology on X; we call it the cocountable topology on X.

(b) Show that a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x \in X$ with respect to the cocountable topology if and only if it is eventually constant with value x, i.e. there exists $N > 0$ such that $x_n = x$ for all $n > N$.

Problem 6. Let X be a topological space. We say that a subset $Z \subseteq X$ is sequentially closed if given any sequence $\{z_n\}_{n\in\mathbb{N}}$ in Z converging to a point $x \in X$, we have $x \in Z$.

(a) Show that if $Z \subseteq X$ is closed, then it is sequentially closed.

(b) Give an example of a topological space X and a subset $Z \subseteq X$ such that Z is sequentially closed but is not closed. (Hint: Problem 5.)