18.901, FALL 2024 — HOMEWORK 3

Each part of each main problem is worth 5 points, so this homework is graded out of 50 points. Each part of each bonus problem is worth 0.5 additional points.

Main problems

Problem 1. Let $f : X \to Y$ be a function between topological spaces. Given a point $x \in X$, we say that f is continuous at x if for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.

(a) Let $x \in X$ and let $X' \subseteq X$ be a neighborhood of x. Regard X' as equipped with the subspace topology. Show that the function f is continuous at x if and only if the restricted function $f|_{X'} : X' \to Y$ is continuous at x.

(b) Show that the function f is continuous if and only if, for every point $x \in X$, the function f is continuous at x.

Problem 2. Let $f : X \to Y$ and $g : Y \to Z$ be two functions between topological spaces.

(a) Show that if f and g are embeddings, then the composition $g \circ f$ is an embedding.

(b) Show that if f and g are quotient maps, then the composition $g \circ f$ is a quotient map.

Problem 3. Let $\{i_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in A}$ be a collection of functions between topological spaces. Let $X := \prod_{\alpha \in A} X_\alpha$ and let $Y := \prod_{\alpha \in A} Y_\alpha$, and equip each of these with the product topology. Let $i : X \to Y$ be the function given by $i((x_\alpha)_{\alpha \in A}) := (i_\alpha(x_\alpha))_{\alpha \in A}$, i.e. the product of the functions i_α.

(a) Show that if each i_α is continuous, then i is continuous.

(b) Show that if each i_α is a surjective open map, then i is a surjective open map.

(c) Show that if each i_α is an embedding, then i_α is an embedding.

Problem 4. In this problem, we regard $\mathbb{R}^n$ as equipped with its standard topology, and any subset of $\mathbb{R}^n$ as equipped with the subspace topology. Let $I := [0, 1] \subset \mathbb{R}$.

(a) Let I_0, I_1, I_2 denote three copies of I. For $k \in \{0, 1, 2\}$, let $g_k : I_k \to I$ be the continuous function defined by $g_k(t) := \frac{k+t}{3}$. Let $g : I_0 \amalg I_1 \amalg I_2 \to I$ be the continuous function whose restriction to the factor I_k is g_k. Show that g is a quotient map.

(b) Let $T \subset \mathbb{R}^2$ be any triangle (meaning a subset consisting of the points lying on the edges of a triangle in the plane). Show that there is a quotient map $f : I_0 \amalg I_1 \amalg I_2 \to T$ (where I_0, I_1, I_2 are again three copies of I).

(c) Let $T \subset \mathbb{R}^2$ be any triangle and let $C \subset \mathbb{R}^2$ be any circle. Show that T and C are homeomorphic.

Problem 5. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces and let X be the product set $\prod_{\alpha\in A} X_\alpha$.

(a) Let $\mathcal{B} \subset \mathcal{P}(X)$ consist of the subsets $\prod_{\alpha\in A} U_\alpha \subseteq X$ where U_α is an open subset of X_α for each $\alpha \in A$. Show that $\mathcal{B}$ is a basis for a topology on X.

The topology generated by the basis $\mathcal{B}$ is called the box topology on the product set X.

(b) Show that the box topology on X is finer than the product topology on X, and is equal to the product topology if the set A is finite.

(c) We now consider the case where $A = \mathbb{N}$ and $X_\alpha = \mathbb{R}$ (equipped with the standard topology) for each $\alpha \in \mathbb{N}$, so that $X = \prod_{\alpha\in\mathbb{N}} \mathbb{R} = \mathbb{R}^{\mathbb{N}}$ is the set of sequences in $\mathbb{R}$. Let $\delta : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ be the function sending x to the constant sequence $(x, x, \ldots)$. Show that f is *not* continuous with respect to the box topology on $\mathbb{R}^{\mathbb{N}}$.