18.901, FALL 2024 — HOMEWORK 4

Each part of each main problem is worth 5 points, so this homework is graded out of 50 points. Each part of each bonus problem is worth 0.5 additional points.

Main problems

Problem 1. In this problem we equip $\mathbb{R}$ with the lower limit topology. Let J denote the interval $[0,1] \subset \mathbb{R}$ equipped with the subspace topology.

(a) Is J compact? (Show why it is or is not.)

(b) Is J Hausdorff? (Show why it is or is not.)

Problem 2. Let X be any set equipped with the cofinite topology.

(a) Show that X is compact.

(b) Suppose that X is infinite. Show that there exists a subset Y of X that is not closed but is compact with respect to the subspace topology.

Problem 3. Let $f : X \to Y$ be a function between topological spaces. We say that f is a closed map if f is continuous and moreover, for every closed subset Z of X, its image $f(Z)$ is a closed subset of Y.

(a) Suppose that f is surjective and a closed map. Show that f is a quotient map.

(b) Suppose that f is surjective and continuous, X is compact, and Y is Hausdorff. Show that f is a closed map.

Problem 4. Let X be a Hausdorff topological space, and let Z_1 and Z_2 be disjoint compact subspaces of X. Show that there exist disjoint open subsets U_1 and U_2 of X such that $Z_1 \subseteq U_1$ and $Z_2 \subseteq U_2$.

Problem 5. Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of Hausdorff topological spaces. Let X be their product $\prod_{\alpha \in A} X_\alpha$ (equipped with the product topology). Show that X is Hausdorff.

Problem 6. In this problem we equip $\mathbb{R}$ with the standard topology and subsets of $\mathbb{R}$ with the subspace topology. Let A_0 denote the interval $[0,1] \subset \mathbb{R}$, and for $n \geq 1$, inductively define

$$A_n := A_{n-1} \cap \left[[0,1] \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \right].$$

For example, A_1 is obtained from A_0 by removing $(\frac{1}{3}, \frac{2}{3})$; and then A_2 is obtained from A_1 by further removing $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$; and so on. Then define

$$A := \bigcap_{n \in \mathbb{N}} A_n.$$

(a) Show that, for each $n \in \mathbb{N}$, the subset A_n is a union of finitely many disjoint closed intervals, and show that A is compact.

(b) Equip the set $\{0,1\}$ with the discrete topology and let B be the product $\prod_{n \in \mathbb{N}} \{0,1\}$ of countably infinitely many copies of $\{0,1\}$ (equipped with the product topology). Show that there is a homeomorphism $f : A \to B$.

Problem 7. Let $f : X \to Y$ be a function between topological spaces. The graph of f is the subset of the product $X \times Y$ defined by $\Gamma_f := \{(x, y) \in X \times Y : y = f(x)\}$.

(a) Suppose that f is continuous. Show that Γ_f is homeomorphic to X.

(b) Suppose that Y is Hausdorff and f is continuous. Show that Γ_f is closed in $X \times Y$.

(c) Suppose that Y is compact Hausdorff and that Γ_f is closed in $X \times Y$. Show that f is continuous.

Problem 8. Let X be a topological space and let U and V be open subspaces of X which cover X. Show that X is homeomorphic to $U \amalg_{U \cap V} V$, the latter being the gluing of U and V along the inclusions of their intersection $U \cap V$ into each.