18.901, FALL 2024 — HOMEWORK 5

Each part of each main problem is worth 5 points, so this homework is graded out of 50 points. Each part of each bonus problem is worth 0.5 additional points.

Main problems

Problem 1. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of path connected topological spaces. Show that the product space $\prod_{\alpha\in A} X_\alpha$ is path connected.

Problem 2. We say that a topological space X is totally disconnected if any nonempty connected subspace of X contains exactly one point.

(a) Let S be a set equipped with the discrete topology. Show that S is totally disconnected.

(b) Regard $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Show that $\mathbb{Q}$ is totally disconnected.

(c) Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of totally disconnected topological spaces. Show that the product space $\prod_{\alpha\in A} X_\alpha$ is totally disconnected.

Problem 3. For n a positive integer and $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, let $\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$.

Let $k \in \mathbb{N}$. Let $\mathrm{S}^k := \{x \in \mathbb{R}^{k+1} : \|x\| = 1\}$ and let $\mathrm{D}^k := \{y \in \mathbb{R}^k : \|y\| \le 1\}$. Let $\sim_1$ be the equivalence relation on S^k generated by the relation $(x_1,\dots,x_k,x_{k+1}) \sim_1 (x_1,\dots,x_k,-x_{k+1})$ for $(x_1,\dots,x_{k+1})$ such that $x_{k+1} \neq 0$. Show that the quotient space $\mathrm{S}^k/{\sim_1}$ is homeomorphic to D^k.

Problem 4. Let S^1 be as in Problem 3. Let $f : \mathrm{S}^1 \to \mathbb{R}$ be a continuous function. Show that there exists $x \in \mathrm{S}^1$ such that $f(x) = f(-x)$.

Problem 5. Let k be a positive integer. Let $\sim_2$ be the equivalence relation on $\mathbb{R}^{k+1} \smallsetminus \{0\}$ given by $x \sim_2 tx$ for $x \in \mathbb{R}^{k+1} \smallsetminus \{0\}$ and $t \in \mathbb{R} \smallsetminus \{0\}$. The quotient space $(\mathbb{R}^{k+1} \smallsetminus \{0\})/{\sim_2}$ is denoted $\mathbb{RP}^k$ and called k-dimensional real projective space. The equivalence class of $(x_0,x_1,\dots,x_k) \in \mathbb{R}^{k+1} \smallsetminus \{0\}$ with respect to $\sim_2$ is denoted by $[x_0 : x_1 : \cdots : x_k] \in \mathbb{RP}^k$.

(a) Show that there is a bijection between $\mathbb{RP}^k$ and the set of lines in $\mathbb{R}^{k+1}$ that pass through the origin.

(b) Show that $\mathbb{RP}^k$ is locally Euclidean of dimension k. (Hint: For each $0 \le i \le k$, consider the subspace of $\mathbb{RP}^k$ consisting of equivalence classes $[x_0 : x_1 : \cdots : x_k]$ such that $x_i \neq 0$.)

(c) Let S^k and D^k be as in Problem 3. Let $\sim_3$ be the equivalence relation on S^k generated by the relation $x \sim_3 -x$ for $x \in \mathrm{S}^k$, and let $\sim_4$ be the equivalence relation on D^k generated by the relation $y \sim_4 -y$ for $y \in \mathrm{S}^{k-1}$. Show that the quotient spaces $\mathrm{S}^k/{\sim_3}$ and $\mathrm{D}^k/{\sim_4}$ are both homeomorphic to $\mathbb{RP}^k$.

(d) Let $\sim_5$ be the equivalence relation on $[0,1]\times[0,1]$ generated by the relations $(t,0) \sim_5 (1-t,1)$ and $(0,t) \sim_5 (1,1-t)$ for $t \in [0,1]$. Show that the quotient space $([0,1]\times[0,1])/{\sim_5}$ is homeomorphic to $\mathbb{RP}^2$.

Problem 6. For this problem you may assume Tychonoff's theorem. Equip the set $\{0,1\}$ with the discrete topology and equip the product $\prod_{n\in\mathbb{N}}\{0,1\}$ with the product topology. Show that there is a quotient map $p : \prod_{n\in\mathbb{N}}\{0,1\} \to [0,1]$.

Problem 7. Recall that given two embeddings of topological spaces $i : A \to X$ and $j : A \to Y$, we defined the associated gluing $X \amalg_A Y$ to be the quotient space of the disjoint union $X \amalg Y$ by the equivalence relation $\sim$ generated by the relation $i(a) \sim j(a)$ for $a \in A$. Let us note now that this definition makes sense even when i and j are arbitrary continuous functions, not necessarily embeddings.

This problem is a continuation of Problem 5.

(a) Let $i : \mathbb{R}^{k+1} \smallsetminus \{0\} \to \mathbb{R}^{k+1}$ be the inclusion function, and let $j : \mathbb{R}^{k+1} \smallsetminus \{0\} \to \mathbb{RP}^k$ be the quotient function for the equivalence relation $\sim_2$. Let $i' : \mathrm{S}^k \to \mathrm{D}^{k+1}$ be the inclusion function, and let $j' : \mathrm{S}^k \to \mathbb{RP}^k$ be the quotient function for the equivalence relation $\sim_3$ (using the homeomorphism of Problem 5(c)). Show that the associated gluings

$$\mathbb{R}^{k+1} \amalg_{\mathbb{R}^{k+1}\smallsetminus\{0\}} \mathbb{RP}^k, \qquad \mathrm{D}^{k+1} \amalg_{\mathrm{S}^k} \mathbb{RP}^k$$

are homeomorphic to each other.

(b) Show that the two gluings in the previous part are both homeomorphic to $\mathbb{RP}^{k+1}$.