18.901, FALL 2024 — HOMEWORK 6

Each part of each main problem is worth 5 points, so this homework is graded out of 40 points.

Main problems

Problem 1. Let X and Y be topological spaces and assume that Y is nonempty. We say that a continuous map $f : X \to Y$ is nullhomotopic if it is homotopic to a constant map.

(a) Show that the the following conditions are equivalent:

 (i) The identity map $\mathrm{id}_Y : Y \to Y$ is nullhomotopic.

 (ii) Y is homotopy equivalent to a topological space with one point.

 We say that Y is contractible if these equivalent conditions are satisfied.

(b) Suppose that Y is contractible. Show that then any continuous map $f : X \to Y$ is nullhomotopic.

Problem 2. A topological group is a group G equipped with a topology (on its underlying set) such that the multiplication map $G \times G \to G$, sending $(a, b) \mapsto ab$, and the inversion map $G \to G$, sending $a \mapsto a^{-1}$, are both continuous.

(a) Let G be a group equipped with a topology. Show that G is a topological group if and only if the map $G \times G \to G$ sending $(a, b) \mapsto ab^{-1}$ is continuous.

(b) Let G be a topological group and let $a, a' \in G$. Show that there exists a homeomorphism $f : G \to G$ such that $f(a) = a'$.

Problem 3. Let (X, x_0) and (Y, y_0) be pointed topological spaces. Let $A := \{a\}$ be the topological space with one point labelled a, and let $i : A \to X$ and $j : A \to Y$ be the embeddings given by $i(a) := x_0$ and $j(a) := y_0$. The wedge sum $X \vee Y$ is defined to be the gluing $X \amalg_A Y$ of X and Y along the embeddings i and j of A.

(a) Let $\mathrm{S}^1 \subset \mathbb{R}^2$ be the unit circle centered at the origin, equipped with the basepoint $(1, 0)$. Show that there is an embedding $f : \mathrm{S}^1 \vee \mathrm{S}^1 \to \mathbb{R}^2$.

(b) Let $B \subset \mathrm{I} \times \mathrm{I}$ be the boundary of the square. Show that there is a quotient map $p : B \to \mathrm{S}^1 \vee \mathrm{S}^1$.

(c) Let $T := \mathrm{S}^1 \times \mathrm{S}^1$. Show that for any two points $t, t' \in T$, there exists a homeomorphism $g : T \to T$ such that $g(t) = t'$.

(d) Let T be as in the previous part and let $t \in T$. Show that there is a subspace U of T that is homeomorphic to $\mathrm{S}^1 \vee \mathrm{S}^1$ and is a deformation retract of $T \setminus \{t\}$.