18.901, FALL 2024 — HOMEWORK 7

Each part of each main problem is worth 5 points, so this homework is graded out of 45 points.

Main problems

Problem 1. Let G be a set with two elements.

(a) Define a group structure on G.

(b) Equip G with the group structure defined in the previous part and let H be any group. Show that the set of group homomorphisms $\phi : G \to H$ is in bijection with the subset $\{h \in H : h^2 = e\}$ of H.

Problem 2. Let X be a topological space. Let $\mathrm{Homeo}(X)$ denote the set of homeomorphisms $f : X \to X$.

(a) Show that there is a group structure on $\mathrm{Homeo}(X)$ where the group operation is given by composition of functions.

For G a group, a continuous action of G on X is a group homomorphism $\phi : G \to \mathrm{Homeo}(X)$, where $\mathrm{Homeo}(X)$ is regarded as a group as in the previous part.

(b) Let G be a group and let $\phi : G \to \mathrm{Homeo}(X)$ be a continuous action of G on X. For $x, y \in X$, write $x \sim_\phi y$ if there exists $g \in G$ such that $y = \phi(g)(x)$. Show that this defines an equivalence relation on X.

Given a continuous action ϕ of a group G on X, we write X/G for the quotient space of X by the equivalence relation $\sim_\phi$ of the previous part (the notation is abusive: the quotient depends not just on X and G, but on the action ϕ).

Problem 3. Let G be a group with two elements. Let B be the quotient $(\mathrm{I} \times \mathrm{I})/\sim$, where $\sim$ is the equivalence relation generated by the relation $(0, t) \sim (1, 1 - t)$ for $t \in \mathrm{I}$ (this is the Möbius band). Let C be the cylinder $\mathrm{S}^1 \times \mathrm{I}$. Show that there is a continuous action of G on C such that the quotient space C/G is homeomorphic to B.

Problem 4. (a) Let G and H be groups. Show that there is a group structure on the product set $G \times H$ where the group operation is given by the formula $(g, h) \cdot (g', h') = (g \cdot g', h \cdot h')$. We refer to this as the product group structure on $G \times H$.

(b) Let X and Y be topological spaces, and let $x_0 \in X$ and $y_0 \in Y$. Show that there is a group isomorphism $\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$, where $X \times Y$ is equipped with the product topology and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ is equipped with the product group structure.

(c) Show that the torus $\mathrm{S}^1 \times \mathrm{S}^1$ is not homotopy equivalent to the sphere S^2.

Problem 5. Let X be the topological space. The suspension of X, denoted $\mathrm{S}(X)$, is defined to be the quotient space $(X \times \mathrm{I})/\sim$, where $\sim$ is the equivalence relation generated by the relations $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$. Show that, if X is path connected, then $\mathrm{S}(X)$ is simply connected.