18.901, FALL 2024 — HOMEWORK 9

Each part of each main problem is worth 5 points, so this homework is graded out of 45 points. Each part of each bonus problem is worth 0.5 additional points.

Main problems

Problem 1. Let $p : Y \to X$ be a covering map of topological spaces.

(a) Let $x \in X$. Show that the subspace topology on $p^{-1}(x) \subseteq Y$ is the discrete topology.

(b) Show that p is an open map (and hence a quotient map as long as it is surjective).

Problem 2. Let Γ be a tree, i.e. a graph in which any two vertices are connected by exactly one path. Let Y be the associated topological space (obtained by beginning with the discrete space of vertex points and then appropriately gluing in a copy of I corresponding to each edge).

Fix a vertex point $y_0 \in Y$. For any point $y \in Y$, we have a well-defined distance $d(y_0, y)$ from y_0 to y, given by the length of the shortest path γ_y from y_0 to y (regarding each edge as having length 1, say).

(a) Define a function $h : Y \times \mathrm{I} \to Y$ as follows: for $y \in Y$ and $t \in \mathrm{I}$, let $h(y, t)$ be the point on the path γ_y with distance $t \cdot d(y_0, y)$ from y_0. Show that h is continuous.

(b) Show that Y is contractible.

Problem 3. Let $S \subseteq \mathbb{R}^2$ be the subset consisting of the edges of the square with vertices $(\pm 1, \pm 1)$. Let $r \in \mathrm{Homeo}(S)$ be the homeomorphism given by rotation around the origin by the angle $\pi/2$, and let $f \in \mathrm{Homeo}(S)$ be the homeomorphism given by reflection over the y-axis. Let $G := \{e, r, r^2, r^3, f, rf, r^2 f, r^3 f\} \subseteq \mathrm{Homeo}(S)$.

(a) Show that $rf = fr^{-1}$.

(b) Show that G is a subgroup of $\mathrm{Homeo}(S)$.

Let $a, b \in \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1, 1)$ be the two generating loops. We know that $\pi_1(\mathrm{S}^1 \vee \mathrm{S}^1, 1)$ is free on $\{a, b\}$, so there is a unique homomorphism $\phi : \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1, 1) \to G$ such that $\phi(a) = r$ and $\phi(b) = f$.

(c) Draw a covering map $p : Y \to \mathrm{S}^1 \vee \mathrm{S}^1$ such that, for some choice of basepoint $y_0 \in p^{-1}(1)$, the subgroup $\mathrm{im}(p_*) \subseteq \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1, 1)$ is equal to the subgroup $\ker(\phi) = \phi^{-1}(e)$.

(d) Let $H := \{e, f\} \subset G$ (note that this is a subgroup). Draw a covering map $p' : Y' \to \mathrm{S}^1 \vee \mathrm{S}^1$ such that, for some choice of basepoint $y_0' \in (p')^{-1}(1)$, the subgroup $\mathrm{im}(p'_*) \subseteq \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1, 1)$ is equal to the subgroup $\phi^{-1}(H)$.

(e) For $p : Y \to \mathrm{S}^1 \vee \mathrm{S}^1$ and $p' : Y' \to \mathrm{S}^1 \vee \mathrm{S}^1$ as in the previous two parts, calculate the sizes of $\mathrm{Aut}(Y/\mathrm{S}^1 \vee \mathrm{S}^1)$ and $\mathrm{Aut}(Y'/\mathrm{S}^1 \vee \mathrm{S}^1)$.

Problem 4. Let G be a free group on the two-element set $\{a, b\}$ and let $n \in \mathbb{N}$ with $n > 2$.

(a) Show that there exists a subgroup $H \subseteq G$ and a subset $S \subseteq H$ such that $|S| = n$ and H is free on S. Hint: You may use Problem 5 below.

(b) Make this explicit: for some choice of S satisfying the previous part, write out the n elements of S as words in a and b.

Problem 5. Let $\Gamma = (V, E)$ be a finite, connected graph, and let Y be the associated topological space. Choose a vertex point $y_0 \in Y$. Show that $\pi_1(Y, y_0)$ is free on a set of n generators, where $n = |E| - |V| + 1$.

Hint: Let Γ' be a maximal subtree of Γ and let $Y' \subseteq Y$ be the associated subspace. You may use without proof that the quotient map $q : Y \to Y/Y'$ is a homotopy equivalence (or you may prove this if you like).