18.901, FALL 2024 — MIDTERM EXAM

Each problem is worth 10 points, so the exam is graded out of 40 points.

Problem 1. Let X be the set $\mathbb{N} \amalg \{\infty\}$, i.e. the set obtained by adjoining one element labelled ∞ to $\mathbb{N}$. Define $\mathcal{T} \subset \mathcal{P}(X)$ to consist of those subsets U of X such that one of the following conditions holds:

(a) $U \subseteq \mathbb{N}$ (i.e $\infty \notin U$);

(b) $\infty \in U$ and $X \setminus U$ is finite.

Show that there exists a metric d on X such that the topology induced by d is equal to $\mathcal{T}$.

Problem 2. Let X and Y be topological spaces. Suppose that there exist embeddings $f : X \to Y$ and $g : Y \to X$. Are X and Y then necessarily homeomorphic? (Justify your answer.)

Problem 3. Let X and Y be compact Hausdorff topological spaces. Let $x_0 \in X$ and $y_0 \in Y$, and suppose that the subspace $X \setminus \{x_0\}$ of X is homeomorphic to the subspace $Y \setminus \{y_0\}$ of Y. Show that X and Y are homeomorphic.

Problem 4. Let X be a connected, T_1, normal topological space. Show that if X has more than one point, then X must have uncountably many points.