Active learning Session 1

We learn about comparing two topologies on a given set and then discuss the interior, the closure and boundary of a subset A of a topological space X.

Problem 1

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X with $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$ and that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$.

Illustrate these concepts by comparing the trivial and discrete topology on any set X and then by comparing the 4 topologies on $X = \{1, 2\}$.

Problem 2

Recall the definition of a neighborhood of a point. Let A be a subset of a topological space X. Define the *interior* of A (denoted $\mathring{A}$), and the *closure* of A (denoted $\overline{A}$) :

$$\mathring{A} = \{x \in X \mid A \text{ is a neighborhood of } x \in X\},$$
$$\overline{A} = \{x \in X \mid X \setminus A \text{ is not a neighborhood of } x \in X\}.$$

Prove that $\mathring{A} \subset A \subset \overline{A}$. Note that the *boundary* of A is defined as $\partial A = \overline{A} \setminus \mathring{A}$.

Problem 3

Prove the equalities

$$\mathring{A} = \bigcup_{U \text{ open},\ U \subset A} U, \qquad \text{and} \qquad \overline{A} = \bigcap_{C \text{ closed},\ C \supset A} C.$$

Problem 4

Use Problem 3 to prove that $\mathring{A}$ is the largest open set contained in A and that $\overline{A}$ is the smallest closed set containing A. Deduce that A is open if and only if $A = \mathring{A}$ if and only if it is a neighborhood of each of its points. [1]

Illustrate the concepts from Problem 2 for $A = [0, 1)$ and $X = \mathbb{R}$.

1. The fact that "A is open if and only if it is a neighborhood of each of its points" was mentioned in class.