Active learning Session 3

This time, we will learn about sequentially closed sets and sequentially continuous maps. The goal is to understand how they relate to the more familiar notions of closed sets and continuous maps.

Definition 1. Let X be a topological space.

— A subspace $A \subset X$ is *sequentially closed* if for any a sequence (a_n) in A with $a_n \to x$ for some $x \in X$, one has $x \in A$.

— A map $f\colon X \to Y$ between topological spaces is *sequentially continuous at* $x \in X$ if for every sequence (x_n) in X with $x_n \to x$, one has $f(x_n) \to f(x) \in Y$. The map f is *sequentially continuous* if it is sequentially continuous at every $x \in X$.

Problem 1

Study the notion of being sequentially closed for the discrete and trivial topology. Is $(0, 1] \subset \mathbb{R}$ sequentially closed in $\mathbb{R}$ with the standard topology ?

Problem 2

Prove that closed subsets are sequentially closed.

Problem 3

Prove that continuous functions are sequentially continuous. More precisely, prove that if $f\colon X \to Y$ is continuous at $x \in X$, then it is sequentially continuous at $x \in X$.

Problem 4

Prove that metric spaces are first countable. [1]

Mini-Lecture

We will state partial converses to the statements of Problems 2 and 3. If time permits we will sketch the idea of the proof.

1. Recall from class that a space X is *first countable* if every $x \in X$ admits a countable basis $\mathscr{B}_x$ of neighborhoods.