Active learning Session 6

Problem 1

Let X be a space, let $A \subset X$ be a subspace and use $i\colon A \to X$ to denote the inclusion map.

— The space X *retracts* onto the subspace $A \subset X$ if there is a continuous map $r\colon X \to A$ (called a *retraction*) such that $r \circ i = \mathrm{id}_A$.

— The space X *deformation retracts* onto the subspace $A \subset X$ if there is a homotopy $f_t\colon X \to X$ (called a *deformation retraction*) with $f_0 = \mathrm{id}_X$ and $f_1\colon X \to A$ a retraction.

Prove that

1. If X deformation retracts onto A, then X and A are homotopy equivalent.
2. Convex subspace of $\mathbb{R}^n$ deformation retract onto a point (i.e. convex subspaces of $\mathbb{R}^n$ are contractible).
3. $D^2 \setminus \{0\}$ deformation retracts onto $\partial D^2 = S^1$

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Problem 2

1. Prove that if a space X retracts onto a subspace A, then the inclusion $i\colon A \to X$ induces an injective homomorphism $i_*\colon \pi_1(A) \to \pi_1(X)$. [1]
2. Prove that if a space X deformation retracts onto a subspace A, then the inclusion $i\colon A \to X$ induces an isomorphism $i_*\colon \pi_1(A) \to \pi_1(X)$ (prove this without using the fact that homotopy equivalences induce isomorphisms on the fundamental group).

Problem 3

Prove that D^2 does not retract onto its boundary $\partial D^2 = S^1$.

Problem 4

Use the third problem to prove *Brouwer's fixed point theorem* :

"Every continuous map $f\colon D^2 \to D^2$ admits a fixed point."

Mini-Lecture

If time permits we discuss the fact that for a tree T in a connected graph G, the projection $G \to G/T$ is a homotopy equivalence.

1. In this problem we're implictly assuming that the basepoint x_0 lies in $A \subset X$.