Active learning Session 7

Problem 1

Let $\{X_\alpha\}_{\alpha\in A}$ be a family of path-connected spaces and let $x_\alpha \in X_\alpha$ for each $\alpha \in A$. Assume that each x_α is a deformation retract of an open neighbhorhood $U_\alpha \subset X_\alpha$. Use van Kampen's theorem to prove that

$$\pi_1\left(\bigvee_{\alpha\in A} X_\alpha\right) = \mathop{\ast}_{\alpha\in A} \pi_1(X_\alpha).$$

Here $\bigvee_{\alpha\in A} X_\alpha$ is the quotient of the disjoint union $\bigsqcup_{\alpha\in A} X_\alpha$ in which all the x_α are identified to a point. Hint : generalise what we did in class for $S^1 \vee S^1$. Bigger hint : look at Example 1.21 of Hatcher's textbook.

Problem 2

Read the following statement and give examples : " If X is a connected graph with maximal tree T, then $\pi_1(X)$ is free on the number of edges of $X \setminus T$." Here recall that a *tree* is a contractible graph, and that a tree in a graph X is *maximal* if it contains all the vertices of X

Mini-Lecture

Depending on time, we'll describe one or two proofs of the assertion in the second problem.

Problem 3

In van Kampen's theorem, prove that the condition on $A_\alpha \cap A_\beta$ being path-connected is necessary ; also prove that the condition on A_α and A_β being open is necessary. In both cases, consider the circle as a union of intervals. [1]

Problem 4

Calculate the fundamental group of the space obtained from the torus $T^2 = S^1 \times S^1$ by attaching a 2-cell to one of the S^1-factors. What if we attach a 2-cell to each S^1-factor ?

1. The condition on the triple intersections being path-connected is necessary. The curious reader might consult p.44 of Hatcher's textbook on algebraic topology.