Active learning Session 9

Problem 1

Consider the covering maps $p\colon \mathbb{R} \to S^1, t \mapsto e^{2i\pi t}$ and $p\colon \mathbb{R}^2 \to S^1 \times S^1 = T^2$. Lift the cell structure on S^1 with one 0-cell and one 1-cell to a cell structure on $\mathbb{R}$. Same question for the "usual" cell structure on the torus T^2 and $\mathbb{R}^2$.

How would you prove that a covering space of an n-dimensional CW complex is again an n-dimensional CW complex. No need to write a proof, just describe an idea.

Problem 2

If a space X admits the structure of an dimensional CW complex, then its *Euler characteristic* can be defined as

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \#\{i\text{-cells of } X\}.$$

We take it for granted that $\chi(X)$ depends only on X and not on the choice of the cell structure.

- Calculate the Euler characteristic of the circle, of the torus and more generally of the surface of genus g.
- Using the first exercise, prove that if $\widetilde{X} \to X$ is a degree n cover (with $n < \infty$), then $\chi(\widetilde{X}) = n\chi(X)$. Verify this formula on the cover $S^2 \to \mathbb{R}P^2$ that was discussed during class.
- Calculate the Euler characteristic of the Klein bottle using that it covers the torus (recall Active learning 8). Verify this using the usual cell decomposition of the Klein bottle.

Problem 3

Use the first exercise (with $n = 1$) to prove that a subgroup of a free group is free.

Mini-Lecture

Following Example 0.4 of Hatcher's book, we describe a CW complex structure on the real projective space $\mathbb{R}P^n$.

Problem 4

Calculate $\pi_1(\mathbb{R}P^n)$.

Problem 5

If time permits : prove that for every group G, there exists a 2-complex X_G with $\pi_1(X_G) = G$.