Problem Set 1

Topological spaces and continuous maps

Solve the following problems, unless the instructions ask you to only solve a subset of them. E.g. in Problem 1, you are to solve two subproblems whereas in Problem 7, you are to solve all the subproblems.

Problem 1

Let X, Y be sets, let $f\colon X \to Y$ be a map, let $A \subset X$ be a subset, let $\{A_i\}_{i\in I}$ be a family of subsets of X, let $B \subset Y$ be a subset and let $\{B_i\}_{i\in I}$ be a family of subsets of Y.

Read each of the following properties of images and inverse images and **prove <u>two</u> of them**.

1. One has the inclusion $A \subset f^{-1}(f(A))$ with equality if f is injective. Similarly, one has the inclusion $f(f^{-1}(B)) \subset B$ with equality if f is surjective.
2. Here are how images interact with unions and intersections : one has the equality $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$ as well as $f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f(A_i)$ with equality if f is injective.
3. For inverse images, unions and intersections, there are the following equalities $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$ and $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$.
4. Here is how images and inverse images interact with complements : one has the equality $f^{-1}(B_i \setminus B_j) = f^{-1}(B_i) \setminus f^{-1}(B_j)$ as well as the inclusion $f(A_i) \setminus f(A_j) \subset f(A_i \setminus A_j)$, with equality if f is injective.
5. Here is how inverse images interact with composition of maps : if $g\colon Y \to Z$ is a map and $C \subset Z$ is a subset, then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

Problem 2

Solve **<u>one</u>** of the two following problems :

1. Prove that in the definition of a topology, the condition "if $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$" is equivalent to the condition "if $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \ldots \cap U_n \in \mathcal{T}$".
2. Let $\mathcal{T}$ be a family of subsets of X, whose elements we call open sets. Additionally, we call a subset $C \subset X$ closed if $X \setminus C$ is open. Prove that $\mathcal{T}$ is a topology if and only if the three following conditions hold :
   (a) the empty set $\emptyset$ and X are closed ;

(b) if $\{C_i\}_{i \in I}$ is a collection of closed sets, then the intersection $\bigcap_{i \in I} C_i$ is also a closed set ;

(c) if C_1 and C_2 are closed, then their union $C_1 \cup C_2$ is also closed.

Problem 3

1. For a set X, prove that the family of subsets

$$\mathcal{T}_f = \{U \subset X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$$

defines a topology. This topology is called the *cofinite topology.*

2. If $(X, \mathcal{T})$ is a topological space and $A \subset X$ is a subset, prove that

$$\mathcal{T}_A = \{U \subset A \mid \text{ there exists an open set } V \subset X \text{ with } U = A \cap V\}$$

defines a topology on A, called *the induced topology.*

Problem 4

Let X, Y be topological spaces, and let $f \colon X \to Y$ be a map.
Solve **one** of the two following problems :

1. Prove that f is continuous if and only if $f^{-1}(C) \subset X$ is closed for every closed set $C \subset Y$.
2. Assume that f is bijective. Prove that f^{-1} is continuous if and only if $f(U) \subset Y$ is open for every open set $U \subset X$. [1]

Problem 5

Let X, Y, Z be topological spaces, and let $f \colon X \to Y$ be a continuous map. Prove **one** of the following statements :

1. If $A \subset X$ is a subspace, then the inclusion $j \colon A \to X$ is continuous.
2. The restriction $f|_A \colon A \to Y$ is continuous.
3. If $B \subset Y$ is a subspace with $\mathrm{im}(f) \subset B$, then the map $g \colon X \to B$ obtained from f by restricting the target is continuous.
4. If $Y \subset Z$ is a subspace, then the function $h \colon X \to Z$ obtained from f by extending the target is continuous.

Problem 6

1. Let $S^1 \subset \mathbb{R}^2$ denote the unit circle, and let $S \subset \mathbb{R}^2$ be the square with vertices $(-1,-1), (1,-1), (1,1), (-1,1)$. Prove that S^1 and S are homeomorphic.

1. We say that a map $f \colon X \to Y$ is *open* if the image of every open set is open.

2. Given $a, b, c, d \in \mathbb{R} \cup \{-\infty, \infty\}$, prove that if $a < b$ and $c < d$, then the intervals (a, b) and (c, d) are homeomorphic.

Problem 7

The definition of a group was mentioned in p-set 0.

1. Prove that the set $\mathbb{Z}$ of integers is a group under addition.
2. Fix a set X. Prove that the set of all bijections $f \colon X \to X$ is a group under composition. When $X = \{1, 2, \ldots, n\}$, this group is denoted S_n and is called the *symmetric group*.
3. Prove that the set of all homeomorphisms of a topological space X forms a group under composition.