Problem Set 10:
Van Kampen's theorem

Solve problems 1 and 2.

Problem 1

1. Using the description of $\mathbb{R}P^2$ and of the Klein bottle $\mathscr{K}$ as quotients of $[0,1] \times [0,1]/\sim$, show that $\mathbb{R}P^2 \# \mathbb{R}P^2 \cong \mathscr{K}$.
2. For the real projective plane $\mathbb{R}P^2$, calculate $\pi_1(\mathbb{R}P^2 \# \mathbb{R}P^2)$ using van Kampen's theorem (i.e. *not* using the first part of this exercise).

Problem 2

1. Prove that if $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to Y$ are covering spaces, then so is $p_X \times p_Y : \widetilde{X} \times \widetilde{Y} \to X \times Y$.
2. Prove that the covering space of a Hausdorff space is Hausdorff.

Problem 3

This problem is optional and won't be graded : it's simply here to provide extra (optional) practice with van Kampen's theorem.

The 3-sphere S^3 can be written as the union of two solid tori $S^1 \times D^2$ glued along their common boundary : indeed

$$S^3 \cong \partial D^4 \cong \partial(D^2 \times D^2) \cong (D^2 \times S^1) \cup (S^1 \times D^2).$$

Put differently, $S^3 = \big((D^2 \times S^1) \sqcup (S^1 \times D^2)\big) / \sim$, where $(x,y) \in S^1 \times S^1 = \partial(D^2 \times S^1)$ is equivalent to $(x,y) \in S^1 \times S^1 = \partial(S^1 \times D^2)$. A sketch of the situation is depicted below. Use this decomposition and van Kampen's theorem to reprove the fact that $\pi_1(S^3) = 1$.