Problem Set 12:
The classification of covering spaces

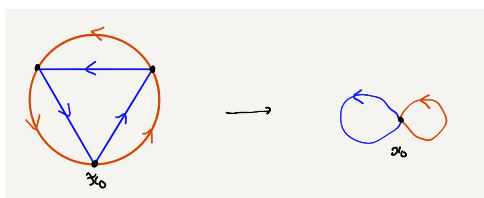
Problem 1

Write the subgroup corresponding to the following covers :

— The cover $S^2 \to \mathbb{R}P^2 = S^2/\sim$ given by the quotient map ;

— The cover $p_n \times p_m\colon S^1 \times S^1 \to S^1 \times S^1, (z, w) \mapsto (z^n, w^m)$; illustrate the cover and describe a loop that lifts to a loop and a loop that does not.

— The cover of $S^1 \vee S^1$ illustrated below :



Problem 2

Draw the universal cover of $S^1 \vee S^2$ and of $\mathbb{R}P^2 \vee \mathbb{R}P^2$. Justify that they are simply-connected and describe the covering map.

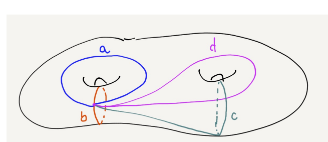
Problem 3

Draw the cover X_H corresponding to the following subgroups $H \leq \pi_1(X, x_0)$; in each case write the degree of the cover :

— For $X = S^1 \vee S^1$ and $H := \langle a^2, b^2, ab\rangle$.

— For $X = \Sigma_2$ $\pi_1(X) = \langle a, b, c, d \mid [a, b][c, d] = 1\rangle$ (as illustrated in the picture below) and $H := \langle a^n ba^{-n}, a^n ca^{-n}, a^n da^{-n} \mid n \in \mathbb{Z}\rangle$.

The trick is to think about the loops that lift and those that don't.



Problem 4

The *deck transformation group* of a cover $p\colon \widetilde{X} \to X$ is the group of homeomorphisms $\varphi\colon \widetilde{X} \to \widetilde{X}$ with $p \circ \varphi = p$ (i.e. the isomorphisms of the cover).

Prove that if $G \curvearrowright Y$ acts properly discontinuously by homeomorphisms on a path-connected space Y, then the deck transformation group of the cover $\pi\colon Y \to Y/G$ is G.

Problem 5

Let G be a group. The *normaliser* of a subgroup $H \leq G$ is defined as

$$N(H) = \{g \in G \mid gHg^{-1} = H\}.$$

Prove that :

1. H is normal in $N(H)$.
2. $N(H) = G$ if and only if H is normal in G.