Problem Set 3

Products and quotients

Solve exercise 6 and **four** of the other five problems.

Problem 1

Endow $\{0,1\}$ with the discrete topology. Prove that $X = \prod_{i \in I}\{0,1\}$ (endowed with the product topology) is discrete if and only if I is finite.

Problem 2

1. Let $\{X_i\}_{i\in I}$ and $\{X_i\}_{i\in J}$ be two families of topological spaces, where the index sets I and J are disjoint. Prove that for the product topology, there is a homeomorphism [1]

$$\prod_{i\in I} X_i \times \prod_{i\in J} X_i \cong \prod_{i\in I\sqcup J} X_i.$$

2. Prove that if $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ are two families of topological spaces, with X_i homeomorphic to Y_i for every $i \in I$, then the products spaces $\prod_{i\in I} X_i$ and $\prod_{i\in I} Y_i$, endowed with the product topology, are homeomorphic.

Problem 3

Prove that the quotient topology is indeed a topology. Additionally, prove that it is the finest topology such that the projection $\pi\colon X \to X/\sim$ is continuous.

Problem 4

In this exercise we endow $\mathbb{R}^n$ with the standard topology and quotients with the quotient topology. Identify the following quotient spaces (i.e. describe a homeomorphism to a simpler space) :

1. $\mathbb{R}/\sim$ where $x \sim x'$ if and only if $\mathrm{sgn}(x) = \mathrm{sgn}(x')$ [2]
2. $\mathbb{R}^2/\sim$ where $(x,y) \sim (z,t)$ if and only if $x = z$.

1. In particular, note that $(X_1 \times \ldots \times X_{n-1}) \times X_n \cong X_1 \times \ldots \times X_n$.
2. The *sign* of x is defined as $\mathrm{sgn}(0) = 0$ and $\mathrm{sgn}(x) = x/|x|$ for $x \neq 0$.

Problem 5

Let $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$ be the unit sphere. Consider $\mathbb{R}P^n = S^n/\sim$ where $x \sim x'$ if and only if $x = -x'$ or $x = x'$, where $-x'$ is the antipode of x. This quotient space is called *real projective space* and for $n = 2$, the *real projective plane.*

1. Find a continuous bijection $\mathbb{R}P^1 \to S^1$.
2. Find a continuous bijection $\mathbb{R}P^2 \to D^2/\sim'$ where $x \sim' y$ if and only if $x = -y$ or $x = y$ for all $x, y \in S^1 = \partial D^2$ (i.e. opposite boundary points of D^2 are identified). [3] In this question, you don't have to write all the details ; an outline is enough.

Using a result from later in the course, both of these continuous bijections will turn out to be homeomorphisms.

Problem 6

Let G be a group. A *subgroup* $H \leq G$ is a non-empty subset $H \subset G$ such that $h_1 h_2^{-1} \in H$ for every $h_1, h_2 \in H$.

1. Let $H \leq G$ be a subgroup. Prove that $e_G \in H$ and if $h \in H$, then $h^{-1} \in H$.
2. Prove that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ by verifying the above definition.
3. Prove that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$ by verifying the above definition.
4. Let G and J be groups. Given a group homomorphism $f\colon G \to J$, prove that
   — $\ker(f) = \{g \in G \mid f(g) = e_J\}$ is a subgroup of G, called the *kernel* of f ;
   — $\operatorname{im}(f) = \{f(g) \in J \mid g \in G\}$ is a subgroup of J, called the *image* of f.
   Use these facts to give a second proof of 2. and 3.
5. Prove that a group homomorphism $f\colon G \to J$ is injective if and only if $\ker(f) = \{e_G\}$.

3. The same statement and proof work in all dimensions, but $n = 2$ is simpler to visualise. Also, note that $D^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ denotes the n-dimensional unit disc.