Problem Set 4
Convergence

Solve the following problems, unless the instructions ask you to only solve a subset of them. E.g. in Problem 1, you are to solve two subproblems whereas in Problem 4, you are to solve all the subproblems.

Problem 1

Solve **two** of the following problems :

1. Assume that S is a subbasis for a topology on a set X and let $x \in X$. Prove that $x_n \to x$ if and only if for every $U \in S$ containing x, there exists an $N > 0$ such that $n \geq N$ implies $x_n \in U$.
2. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and endow $\prod_{i \in I} X_i$ with the product topology. Prove that a sequence (x_n) of elements of $\prod_{i \in I} X_i$ converges to a limit $x \in \prod_{i \in I} X_i$ if and only if for every $i \in I$, the sequence $\pi_i(x_n)$ of i-th coordinates converges to $\pi_i(x) \in X_i$. [1]
3. Is the statement of 2. true if $\prod_{i \in I} X_i$ is endowed with the box topology instead of the product topology ?

Problem 2

Solve **one** of the following problems :

1. Let X be a set endowed with the cofinite topology. Prove that a sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ if and only if for every $y \neq x$, the set $\{n \in \mathbb{N} \mid x_n = y\}$ is finite.
2. For $X = \mathbb{R}$ with the cofinite topology, towards what point(s) does the sequence $x_n = 1/n$ converge ?

Problem 3

Solve **two** of the following problems :

1. Prove that the product of Hausdorff spaces is Hausdorff, both for the product and box topologies.
2. Prove that if X is infinite, then X endowed with the cofinite topology is not Hausdorff.

1. Recall that $\pi_i \colon \prod_{i \in I} X_i \to X_i$ denotes the projection onto the i-th coordinate.

3. Prove that the quotient of a Hausdorff space need not be Hausdorff.

Problem 4

A subgroup $N \leq G$ is *normal* if $gng^{-1} \in N$ for every $g \in G$ and every $n \in N$; the notation is $N \trianglelefteq G$.

1. A group G is *abelian* if $gh = hg$ for every $g, h \in G$. Prove that every subgroup of an abelian group is normal; in particular $n\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$.
2. Prove that $SL_n(\mathbb{Z})$ is a normal subgroup of $GL_n(\mathbb{Z})$.
3. Given a group homomorphism $f\colon G \to H$, prove that the kernel of f is a normal subgroup of G.
4. Prove that if N is a subgroup, then "$g \sim h$ if and only if $gh^{-1} \in N$ for every $g, h \in G$" defines an equivalence relation on G.