Problem Set 5

Connectedness and Path Connectedness

Problem 1

1. Determine when a set X endowed with the cofinite topology is connected. Prove the resulting statement.
2. Prove that if $\sim$ is an equivalence relation on a connected topological space X, then $X/\sim$ is connected.

Problem 2

Prove the following statements.

1. A product of path-connected spaces is path-connected.
2. If $f\colon X \to Y$ is continuous and X is path-connected, then so is $f(X)$.
3. The quotient of a path-connected space is path-connected. [1]

Problem 3

Let X be a topological space.

1. Define a relation on X by $x \sim y$ if and only if there exists a connected subspace $Y \subset X$ that contains both x and y, for every $x, y \in X$. Verify that this is an equivalence relation. The equivalence classes are called the *connected components* of X.
2. Define a relation on X by $x \sim y$ if and only if there exists a path from x to y for $x, y \in X$. Verify that this is an equivalence relation. The equivalence classes are called the *path components of* X.

Problem 4

If $N \triangleleft G$ is a normal subgroup, then we saw in the fourth problem set that "$g \sim h$ if and only if $gh^{-1} \in N$" defines an equivalence relation on G. The *quotient group* of G by N, denoted G/N, consists of the resulting equivalence classes. The goal of this exercise is to understand why G/N is indeed a group and to work out an example.

1. This implies for instance that the torus, the Möbius band, the Klein bottle, the sphere and the real projective plane are all path-connected. Indeed, we described each of these spaces as a quotient of the path-connected space $[0, 1] \times [0, 1]$.

1. We want to define the group law on G/N as $[g]*[h] := [gh]$. Prove that this is independent of the choice of representatives for the classes $[g]$ and $[h]$, i.e. prove that if $g \sim g'$ and $h \sim h'$, then

$$[g] * [h] = [gh] = [g'h'] = [g'] * [h'].$$

2. Prove that $(G/N, *)$ is a group.
3. Prove that the group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the group $\mathbb{Z}_n$.