Problem Set 6:
Compactness

Solve problems 4, 6 and **three** of the other four problems.

Problem 1

1. Prove that a set endowed with the cofinite topology is compact.
2. What about the cocountable topology ?

Problem 2

Are the following properties of compactness correct ? If so, prove them, otherwise give counterexamples.

1. The quotient of a compact space is compact.
2. A finite union of compact subspaces is compact.
3. Every subspace of a compact space is compact.

Problem 3

Let X and Y be subspaces of $\mathbb{R}^n$. For each of the following situations, determine whether X and Y can be homeomorphic :

1. X closed and bounded, Y closed but not bounded.
2. X bounded, Y non-bounded.
3. X closed and bounded, Y bounded but not closed.
4. X closed, Y not closed.

If you think that such X and Y exist, give an example (and justify). Otherwise prove why no such pairs exist.

Problem 4

Using connectedness and compactness, which of the following spaces can you prove are not homeomorphic : $[0,1], \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, S^1, S^2, \Sigma_1 = S^1 \times S^1$? [1]

1. It turns out that no two pairs of these spaces are homeomorphic and developping the tools to study this kind of question will be the goal of the second part of this course.

Problem 5

Are the following properties of sequential compactness correct ? If so, prove them, otherwise give counterexamples :

1. The quotient of a sequentially compact space is sequentially compact.
2. A finite union of sequentially compact spaces is sequentially compact.
3. Every subspace of a sequentially compact space is sequentially compact.
4. The image of a sequentially compact space by a continuous map is sequentially compact.

Problem 6

Let G and H be groups and let $N \triangleleft G$ be a normal subgroup.

1. We say that a homomorphism $f\colon G \to H$ *descends* to G/N whenever $g \sim h$ implies $f(g) = f(h)$ for every $g, h \in G$. In this case, the map *induced by* f on the quotient is $\widetilde{f}\colon G/N \to H, [g] \mapsto f(g)$. Check that $\widetilde{f}$ is a group homomorphism.
2. Prove that any group homomorphism $f\colon G \to H$ induces an isomorphism $\widetilde{f}\colon G/\ker(f) \to \mathrm{im}(f)$; this result is sometimes referred to as the *first isomorphism theorem.*
3. Use the first isomorphism theorem to prove that the exponential map $\mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$ descends to an isomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$
4. Use the first isomorphism theorem to prove that
$$GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R} \setminus \{0\}.$$
5. Consider the normal subgroup $N := \mathbb{Z} \times \{0\} \trianglelefteq \mathbb{Z}^2 =: G$ and find to what group G/N is isomorphic to.

Note that *index* $[G\colon H]$ of a subgroup $H \leq G$ is the cardinality of the set G/H. For instance, $[\mathbb{Z}\colon n\mathbb{Z}] = n$ and $[GL_n(\mathbb{Z})\colon SL_n(\mathbb{Z})] = 2$.