Problem Set 7:
The fundamental group

Solve problem 6 and **four** of the other five problems.

Problem 1

Consider the unit sphere $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\}$ and denote the north pole $(0, \ldots, 0, 1) \in S^n$ by N. Prove that the *stereographic projection*

$$\begin{aligned} S^n \setminus \{N\} &\to \mathbb{R}^n \\ (x_1, \ldots, x_{n+1}) &\mapsto \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right) \end{aligned}$$

is a homeomorphism.

Problem 2

Let X be a topological space and let $x_0 \in X$. Prove that $\pi_1(X, x_0) \cong \pi_1(C(x_0), x_0)$, where $C(x_0) \subset X$ is the path-component of X containing x_0.[1]

Problem 3

Let X be a path-connected topological space, and let $h\colon I \to X$ be a path from $x_0 \in X$ to $x_1 \in X$.

1. Prove that the map

$$\begin{aligned} \beta_h\colon \pi_1(X, x_1) &\to \pi_1(X, x_0) \\ [f] &\mapsto [h \cdot f \cdot \overline{h}]. \end{aligned}$$

   is a group isomorphism.
2. Prove that $\beta_h([f])$ only depends on the homotopy class of h, i.e. if $h_0 \simeq h_1$, then $\beta_{h_0}([f]) = \beta_{h_1}([f])$.

Problem 4

Prove that a topological space X is simply-connected if and only if there is a unique homotopy class of paths connecting any two points in X.

1. Path components were introduced in p-set 5, problem 3.

Problem 5

Given two groups G_1 and G_2, one can endow the product $G_1 \times G_2$ with the law $(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, g_2 h_2)$; this turns $G_1 \times G_2$ into a group (you don't have to check this).

Let X and Y be topological spaces, and let $x_0 \in X$ and $y_0 \in Y$. Prove that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Problem 6

Given a subset $R \subset G$, consider

$$\langle R \rangle = \bigcap_{H \leq G, R \subset H} H.$$

1. Prove that $\langle R \rangle$ is the smallest subgroup of G containing R.
2. Prove that $\langle R \rangle = \{r_1^{\varepsilon_1} \cdots r_n^{\varepsilon_n} \mid r_i \in R,\ \varepsilon_i = \pm 1 \text{ for each } i\}$.
3. We say that G *is generated by* R if $G = \langle R \rangle$. Prove that $\mathbb{Z}$ is generated by $R = \{1\}$ and find a generating set $R \subsetneq S_4$ for the symmetric group S_4.