Problem Set 8:
The fundamental group and homotopy equivalences

Solve problem 2 and **four** of the other five problems.

Problem 1

A topological space is called *contractible* if it is homotopy equivalent to a point. Prove that a topological space X is contractible if and only if id_X is homotopic to a constant map.

Problem 2

In this exercise, it is not expected you write down explicit homeomorphisms and homotopy equivalences.

1. Classify the following letters up to homeomorphism and homotopy equivalence :

   A,B,C,D,E,F,G,H,I,J.

2. Let X, Y be topological spaces, let $x_0 \in X$ and let $y_0 \in Y$. Define the *wedge* (or *one point union*) of X and Y as $X \vee Y = X \sqcup Y/\sim$ where $x_0 \sim y_0$. Explain why the punctured genus g surface is homotopy equivalent to the wedge of $2g$ circles, i.e. given an open ball $B \subset \Sigma_g$, illustrate why $\Sigma_g \setminus B$ is homotopy equivalent to $\bigvee_{i=1}^{2g} S^1$.

Problem 3

In this exercise, it is expected you write down explicit formulas for homotopies and homotopy equivalences.

1. Prove that the cylinder $Cyl = S^1 \times [0,1]$ is homotopy equivalent to a circle.
2. By viewing the Möbius band $\mathcal{M}$ as a quotient of $[0,1]\times[0,1]$, prove that $\mathcal{M}$ is homotopy equivalent to a circle.

Calculate $\pi_1(Cyl)$ and $\pi_1(\mathcal{M})$.

Problem 4

Recall the following notation for the unit n-sphere and the unit $(n+1)$-disc :

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\},$$
$$D^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 \leq 1\}.$$

Write $\iota\colon S^n \to D^{n+1}$ for the inclusion map. Prove that a continuous map $f\colon S^n \to X$ is homotopic to a constant map if and only if there exists a continuous map $\widetilde{f}\colon D^{n+1} \to X$ such that $\widetilde{f} \circ \iota = f$.

Problem 5

Given a subset $R \subset G$, consider

$$\langle\langle R \rangle\rangle = \bigcap_{N \lhd G, R \subset N} N.$$

1. Prove that $\langle\langle R \rangle\rangle$ is the smallest normal subgroup of G containing R.
2. Prove that $\langle\langle R \rangle\rangle = \{g_1 r_1^{\varepsilon_1} g_1^{-1} \cdots g_n r_n^{\varepsilon_n} g_n^{-1} \mid r_i \in R,\ \varepsilon_i = \pm 1,\ g_i \in G\}$.

We say that G *is normally generated by* R if $G = \langle\langle R \rangle\rangle$.

Problem 6

Let X be a set, let F be a group and let $j\colon X \to F$ be a map. Assume that (F, j) satisfies the following *universal property* : for every group G and every map $f\colon X \to G$, there exists a unique homomorphism $\widetilde{f}\colon F \to G$ such that $\widetilde{f} \circ j = f$.[1] Prove that F is isomorphic to the free group $F(X)$.

1. During class, we proved that $(F(X), \iota\colon X \to F(X))$ satisfies this universal property.