Problem Set 9:

Group presentations

Problem 1

Let $\{G_\alpha\}_{\alpha\in A}$ be a family of groups.

1. Let F be a group and let $\{\iota_\alpha\colon G_\alpha \to F\}_{\alpha\in A}$ be a family of homomorphisms. Assume that $(F, \{\iota_\alpha\}_{\alpha\in A})$ satisfies the following *universal property* : for every group G and for every family $\{\varphi_\alpha\colon G_\alpha \to G\}_{\alpha\in A}$ of homomorphisms, there exists a unique homomorphism $\varphi\colon F \to G$ such that $\varphi \circ \iota_\alpha = \varphi_\alpha$ for every α.[1] Prove that $F \cong *_\alpha G_\alpha$.
2. Assume that each group G_α is presented by $\langle X_\alpha \mid R_\alpha\rangle$. Prove that $*_\alpha G_\alpha \cong \langle \sqcup_\alpha X_\alpha \mid \sqcup_\alpha R_\alpha\rangle$ by proving that $\langle \sqcup_{\alpha\in A} X_\alpha \mid \sqcup_{\alpha\in A} R_\alpha\rangle$ satisfies the universal property of 1.
3. Deduce that $F(X) \cong *_{\alpha\in X}\mathbb{Z}$.

Problem 2

1. Prove that $S_3 \cong \langle x, y \mid x^2, y^2, (xy)^3\rangle$.
2. Identify the following group $\langle x, y \mid x^4 = y^3, x^3 = y^4\rangle$.
3. Prove that $\langle a, b \mid a^2 = b^2\rangle$ and $\langle c, d \mid cdcd^{-1}\rangle$ are two presentations for the same group.

Problem 3

Use van Kampen's theorem to calculate the fundamental group of the Klein bottle $\mathscr{K}$ in two different ways :

1. by viewing $\mathscr{K}$ as the union of two Möbius bands along their boundary,
2. by writing $\mathscr{K}$ as the union of a disc D^2 with $\mathscr{K} \setminus \mathrm{Int}(D^2)$.

1. During class, we proved that $(*_{\alpha\in A} G_\alpha, \{j_\alpha\colon G_\alpha \to *_\alpha G_\alpha\}_{\alpha\in A})$ satisfies this universal property.